JUNE 1960

Journal
of
The Franklin Institute

1826

1960

The Thunder-Storm as a Chemical Phenomenon

R. v. KONOW

On Helmholtz's Theorem in Multiply-Bounded and Multiply-Connected Regions

J. VAN BLADEL

Stresses in Certain Thin Plates Rotating about an Axis Lying in Their Middle Plane

R. S. DHALIWAL

Some Considerations of the Mechanics and Design Limitations of Autofrettage

J. H. FAUPEL

VOL. 269 NO. 6

As a Direct-Reading Instrument *The extreme sensitivity of the d-c amplifier is utilized to check plasticizer insulation resistance values in the megamegohm range.*

As a Recorder Preamplifier *The rack-mounted amplifier above is being used to increase the sensitivity of a recorder in running special tests of switches.*

As a Null Detector *The d-c Amplifier is being used above for factory checking and calibration of instruments.*

L&N Low Level d-c Amplifiers are

Triple Purpose... for Triple Service

● In response to the constant demand for versatility in precision instruments, these d-c Indicating Amplifiers combine the functions of three useful instruments in one:

1. A Direct-Reading Instrument that is always ready to use . . . never any readjusting of zero, either initially or during a series of readings. Simply select the range in which you want to work by turning scale-multiplier knob.

2. A Recorder Preamplifier — Values measured by Stabilized d-c Amplifiers can be recorded directly on Speedomax recorders.

3. A Null Detector more sensitive than most reflecting galvanometers, yet with full scale response time of only 2 to 3 seconds. These instruments are unaffected by vibration; leveling is not necessary. At the turn of a range knob, a wide choice of sensitivities can be obtained without external shunts. A non-linear response characteristic is also available for easy balancing.

These amplifiers are suitable for handling low level measurements with thermocouples, strain gages, bolometers—bridge and potentiometer circuits—ionization, leakage and phototube currents—almost any measurement of extremely small direct current or voltage.

Self-contained, the unit can either be used "as is" or removed from case and mounted on a 19" relay rack.

For details, including complete specifications, send for Folder EM9-51(1). Write our nearest office, or 4992 Stenton Ave., Phila. 44, Penna.

LEEDS NORTHRUP

Instruments automatic controls • furnaces

INDEX TO ADVERTISERS

	PAGE
Almo Radio Co.	vii
Atlas Photo Engraving Co.	ix
Bell Telephone Laboratories	x
Buten, M., & Sons	vi
Coleman, W. B., & Co.	iv
Damon & Foster	iv
Davidson, P. L.	iv
Franklin Institute, JOURNAL	iii
Franklin Institute, Laboratories	Fourth cover page
Harris-Dechant Associates	iv
Herbach & Rademan	ix
Hess & Barker	vii
Lancaster Press, Inc.	ii
Leeds & Northrup Co.	Second cover page
Leopold, Charles S.	iv
Levy & Co., Max	vi
Northeast Roofing Supply Co.	ix
Shanahan & Co.	ix
Texas Instruments, Inc.	Third cover page
Thomas Co., Arthur H.	vi
Wiljhelm, C.	vii
Yarnall-Waring Co.	ii
Young, George, Co.	ix

Second-class postage paid at Lancaster, Pa.

YARWAY

STEAM PLANT EQUIPMENT

OVER 50 YEARS A PARTNER
IN THE PROGRESS OF
STEAM ENGINEERING

Yarnall-Waring Company • Philadelphia 18

A COMPLETE PRINTING SERVICE

GOOD PRINTING does not just happen; it is the result of careful planning. The knowledge of our craftsmen, who for many years have been handling details of composition, printing and binding, is at your disposal. For over seventy-five years we have been printers of scientific and technical journals, books, theses, dissertations and works in foreign languages. Consult us about your next job.

. PRINTERS OF .
JOURNAL OF THE
FRANKLIN INSTITUTE

LANCASTER PRESS, Inc.

PRINTERS • BINDERS • ELECTROTYPERS

ESTABLISHED 1877 LANCASTER, PA.

Announcing publication of

The Proceedings of the Symposium on

–GAS-COOLED REACTORS–

held February 10-11, 1960

sponsored jointly by The Franklin Institute and the Delaware Valley Section of the American Nuclear Society.

Twenty-one papers are featured by well known U. S. research people and several foreign scientists who have reviewed the developments in this new type of atomic power plant. The symposium aimed to clarify the problems which must be solved before gas cooling can be used to its fullest potential.

Sir William Cook, C.B., of the U. K. Atomic Energy Authority reviewed the British program in gas-cooled reactors; a representative from the Swiss firm, Brown, Boveri & Cie., discussed their work on a pebble bed reactor which they are building for the West German Government. Representatives of the Australian Atomic Energy Commission and from West Germany presented programs being carried out in their respective countries. Discussions are included at the end of most papers.

The sessions were devoted to the following topics:

I Status of Major Gas-Cooled Reactor Programs

II Fuel Elements and Materials

III Components and Loop Studies

Issued as Monograph No. 7 under the auspices of the JOURNAL OF THE FRANKLIN INSTITUTE, the Proceedings are now available. Send your order (at $5.00 per copy) to

JOURNAL OF THE FRANKLIN INSTITUTE
20th & Parkway
Philadelphia 3, Pa.

Consulting Engineers

HARRIS-DECHANT ASSOCIATES

FREDERICK H. DECHANT
Consulting Engineer

123 So. Broad St.
Philadelphia 9, Pa.

P. L. DAVIDSON
Consulting Engineer

◆

Philadelphia, Pa. and
Greensboro, N. C.

DAMON & FOSTER

Consulting Engineers
Surveyors

CHESTER PIKE & HIGH ST.
SHARON HILL, PA.

CHARLES S. LEOPOLD, INC.
ENGINEERS

◆

215 SOUTH BROAD ST.
PHILA. 7, PA.

W. B. COLEMAN & CO.
Metallurgists - Chemists - Engineers

Spectrographic and Spectrophotometric Analysis
Chemical and Physical Testing—Microscopic Investigation
Huey and Salt Spray Corrosion Tests—Boiler Water Conditioning

9TH & RISING SUN AVE., PHILADELPHIA 40, PA.

THE FRANKLIN INSTITUTE

OFFICERS

President.....WYNN LAURENCE LEPAGE

Executive Vice President.....J. G. RICHARD HECKSCHER
Secretary.....W. F. JACKSON, JR.
Treasurer.....C. M. WATERBURY
Director of Laboratories.....NICOL H. SMITH
Assistant Secretaries.....C. M. WATERBURY, WALTER A. R. PERTUCH
Assistant Treasurer.....JOHN J. JACKSON

BOARD OF MANAGERS

Term expires in 1961	*Term expires in 1962*	*Term expires in 1963*
HENRY B. BRYANS	HENRY B. ALLEN	BRANDON BARRINGER
HENRY M. CHANCE II	G. H. CLAMER	EDWARD G. BUDD, JR.
FRANCIS J. CHESTERMAN	A. FELIX DUPONT, JR.	J. FRANK COX
JAMES CREESE	C. L. JORDAN	JOSEPH GRAY JACKSON
RUPEN EKSERGIAN	LIONEL F. LEVY	JAMES MCGOWAN, JR.
JOSEPH A. FISHER	R. G. RINCLIFFE	RICHARD T. NALLE
MORTON GIBBONS-NEFF	JAMES H. ROBINS	S. WYMAN ROLPH
GAYLORD P. HARNWELL	PHILIP H. WARD, JR.	JOHN RUSSELL, JR.
WYNN LAURENCE LEPAGE	HARLESTON R. WOOD	W. MAXWELL SCOTT, JR.
EMERY W. LOOMIS		I. MELVILLE STEIN

GEORGE WHARTON PEPPER, Honorary Member of the Board

EXECUTIVE COMMITTEE OF THE BOARD OF MANAGERS

S. WYMAN ROLPH, *Chairman*
RICHARD T. NALLE, *Vice Chairman*
HENRY B. BRYANS — JOSEPH A. FISHER — WYNN LAURENCE LEPAGE

STANDING COMMITTEES OF THE BOARD OF MANAGERS

Nominating Committee.....RICHARD T. NALLE, *Chairman*
Budget Committee.....HENRY B. BRYANS, *Chairman*
Personnel Policy Committee.....HARLESTON R. WOOD, *Chairman*
Buildings and Grounds Committee.....W. MAXWELL SCOTT, JR., *Chairman*
Education Committee.....JOSEPH GRAY JACKSON, *Chairman*
Research and Development Committee.....G. H. CLAMER, *Honorary Chairman*
EMERY W. LOOMIS, *Chairman*
Bartol Research Foundation Committee.....G. H. CLAMER, *Honorary Chairman*
GAYLORD P. HARNWELL, *Chairman*
Biochemical Research Foundation Committee.....A. FELIX DUPONT, JR., *Chairman*
Educational Radio and Television Committee.....HENRY B. BRYANS, *Chairman*

GENERAL COMMITTEES OF THE INSTITUTE

Committee on Science and the Arts.....J. PHILIP EVANS, *Chairman*
Membership Committee.....MORTON GIBBONS-NEFF, *Chairman*
Investment Committee.....BRANDON BARRINGER, *Chairman*
Endowment Committee.....MORTON GIBBONS-NEFF, *Chairman*

ADVISORY COMMITTEES OF THE INSTITUTE

Museum and Memorial Committee.....I. MELVILLE STEIN, *Chairman*
Library Committee.....PENROSE R. HOOPES, *Chairman*
Meetings Committee.....LESLIE J. WOODS, *Chairman*
Public Relations Committee.....C. L. JORDAN, *Chairman*
Publications Committee.....LIONEL F. LEVY, *Chairman*
Laboratories Committee.....RUPEN EKSERGIAN, *Chairman*

HOSTESS COMMITTEE OF THE INSTITUTE

MRS. ISAAC A. PENNYPACKER, *Chairman*

PRECISION RULINGS ON GLASS

Scales . Grids . Reticles
Halftone Screens

MAX LEVY & CO. . Wayne Ave. & Berkley St. Phila., Pa.

EVERYTHING IN PAINTS

BUTEN'S

PAINT STORES

Phila., Chester, Reading, Upper Darby, Bryn Mawr, Darby, Doylestown
Camden, N. J. — Wilmington, Del.

ARTHUR H. THOMAS CO.

Laboratory Apparatus and Reagents

Selected for Chemistry and Biology

Adequate stocks of 16,000 Apparatus items and 6,000 Reagent items carried in our Philadelphia warehouse for immediate shipment.

VINE STREET AT THIRD PHILADELPHIA 5, PA.

C. WILJHELM
INDUSTRIAL INSTRUMENTS, INC.
HUNTINGDON VALLEY, PA.
CApital 4-2043

SALE OR RENTAL—PYROMETERS, FLOWMETERS, THERMOMETERS, CO_2 RECORDERS, MOST OTHER TYPES OF INDUSTRIAL INSTRUMENTS & CONTROLS

REPAIR—Competent & Quick Repair of *ALL* Types of Industrial Instruments —at *your* plant or in our laboratory

Almo RADIO CO.
EXECUTIVE OFFICES 913 ARCH ST. PHILA., PA. · WA 2-5918
10 LOCATIONS IN FOUR STATES TO SERVE YOU

DOWNTOWN PHILA.	WEST PHILA.	NORTHEAST	INDUSTRIAL DEPT.
913 ARCH STREET	6205 MARKET ST.	7540 FRANKFORD AVE.	412 N. 6th STREET
WA 2-5918	GR 6-1706	DE 2-2234	WA 2-5918

BRANCHES: CAMDEN, N. J. • ATLANTIC CITY, N. J. • TRENTON, N. J.
NORRISTOWN, PENNA. • WILMINGTON, DELAWARE • SALISBURY, MD.

60 YEARS OF EXPERIENCE
ENGINEERING MACHINISTS
DESIGN · MANUFACTURE · REPAIRS
HESS & BARKER
930 Washington Ave. · HOward 7-1121 · Phila. 47, Pa.

AWARDS BY THE INSTITUTE

The Franklin Medal (1914—Gold Medal).—This medal is awarded annually to those workers in physical science or technology, without regard to country, whose efforts, in the opinion of the Institute, acting through its Committee on Science and the Arts, have done most to advance a knowledge of physical science or its applications.

The Elliott Cresson Medal (1848—Gold Medal).—This medal is awarded for discovery or original research, adding to the sum of human knowledge, irrespective of commercial value; leading and practical utilizations of discovery; and invention, methods or products embodying substantial elements of leadership in their respective classes, or unusual skill or perfection in workmanship.

The Howard N. Potts Medal (1906—Gold Medal).—This medal is awarded for distinguished work in science or the arts; important development of previous basic discoveries, inventions or products of superior excellence or utilizing important principles.

The John Price Wetherill Medal (1925—Silver Medal).—This medal is awarded for discovery or invention in the physical sciences or for new and important combinations of principles or methods already known.

The Edward Longstreth Medal (1890—Silver Medal).—This medal is awarded for inventions of high order and for particularly meritorious improvements and developments in machines and mechanical processes. In the event of an accumulation of the fund for medals beyond the sum of one hundred dollars, it is competent for the Committee on Science and the Arts to offer from such surplus a money premium for some special work on any mechanical or scientific subject that is considered of sufficient importance.

The Louis E. Levy Medal (1923—Gold Medal).—This medal is awarded to the author of a paper of especial merit, published in the JOURNAL OF THE FRANKLIN INSTITUTE, preference being given to one describing the author's experimental and theoretical researches in a subject of fundamental importance.

The George R. Henderson Medal (1924—Gold Medal).—This medal is awarded for meritorious inventions or discoveries in the field of Railway Engineering.

The Walton Clark Medal (1926—Gold Medal).—This medal is awarded to the "author of the most notable advance in knowledge or improvement in apparatus, or in method concerning the science or the art of gas manufacture or distribution or utilization in the production of illumination, or of heat, or of power."

The Frank P. Brown Medal (1938—Silver Medal).—This medal is awarded to inventors for discoveries and inventions involving meritorious improvements in the building and allied industries.

The Newcomen Medal (1943—Gold Medal).—This medal is awarded, not oftener than once in three years, for achievement in the field of Steam.

The Francis J. Clamer Medal (1943—Silver Medal).—This medal is awarded at least once in five years for meritorious achievement in the field of Metallurgy.

The Stuart Ballantine Medal (1946—Gold Medal).—This medal is to be awarded in recognition of outstanding achievement in the fields of Communication and Reconnaissance which employ electromagnetic radiation.

The Boyden Premium (1859).—This premium is awarded not oftener than once in five years to any resident of North America who has recently made a notable experimental determination of the speed, in free space, of radiation in any region of the entire spectrum.

The William M. Vermilye Medal (1937—Bronze Medal).—This medal is awarded not oftener than biennially in recognition of outstanding contribution in the field of Industrial Management.

The Certificate of Merit (1882).—A Certificate of Merit is awarded to persons adjudged worthy thereof for meritorious inventions, discoveries or improvements in physical processes or devices.

For further information relating to these awards apply to the Secretary, Committee on Science and the Arts

Commercial Stationery
Loose Leaf—Blank Books
Filing Equipment
Office Supplies

SHANAHAN & CO.
22 S. 18 St. Ri 6-0333

IS THE ANSWER
ATLAS PHOTO ENGRAVING CO. 1208 CHERRY ST. PHILA., 7, PA.

HEADQUARTERS
FOR
RADIO PARTS ELECTRONIC COMPONENTS
GEIGER-MÜLLER COUNTER TUBES
AND
RADIO ACTIVITY DETECTORS

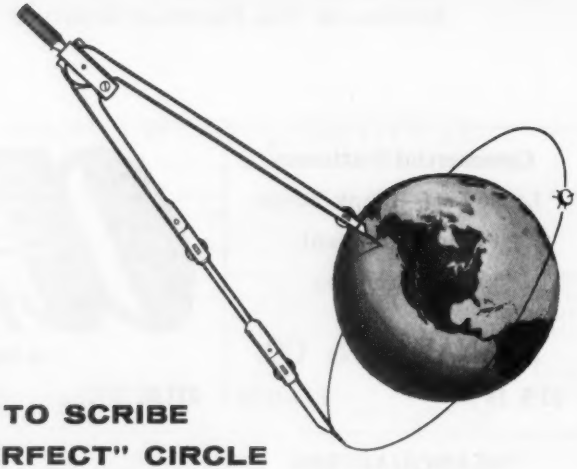
HERBACH & RADEMAN
1204 ARCH STREET PHILADELPHIA 7, PA.

GEORGE YOUNG COMPANY

RIGGING AND HAULING
CONTRACTORS

S. W. COR. 20th & OREGON AVE.
PHILADELPHIA 45, PA. HO 7-5315

INDUSTRIAL COATINGS, ADHESIVES
AND MAINTENANCE PRODUCTS
—a specialized service—
NORTHEAST ROOFING SUPPLY COMPANY
Industrial Division
2nd Street at Fishers Avenue Philadelphia 20, Pa.
GLadstone 5-5500 GLadstone 5-5500
Delaware Valley distributors for all products of THE FLINTKOTE COMPANY

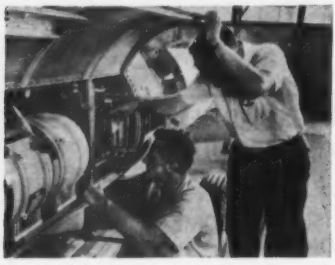


HOW TO SCRIBE A "PERFECT" CIRCLE IN OUTER SPACE

Bell Laboratories guidance system achieves unprecedented accuracy in steering Tiros weather satellite into orbit

Equipped with TV cameras, tape recorders, solar cells and antennas, the world's most advanced weather satellite, the NASA Tiros I, had to be placed in a precisely circular orbit at a specified altitude to do its job well.

The "shot" was a virtual bull's-eye. The mean altitude was within *one mile* of that planned. And the deviation from this mean was less than ½ per cent, making it the most-nearly-perfect circular orbit ever achieved with a space vehicle by either the United States or Russia.



Two Bell Laboratories engineers, T. J. Grieser and D. R. Hagner, look over the second-stage section of the Air Force Thor-Able missile used to launch the NASA Tiros weather satellite.

The dependability and accuracy of Bell Laboratories' ground-controlled Command Guidance System has been proven before—in the successful test flights of the Air Force Titan intercontinental ballistic missile, and in last year's Air Force Thor-Able reentry test shots from which the first nose-cone recoveries were made at ICBM distance. Now, with Tiros, the system serves a dramatic *non*-military project. Other uses are in the offing.

This achievement in precise guidance again illustrates the versatility of Bell Laboratories' research and development capabilities—directed primarily toward improving your Bell telephone service.

BELL TELEPHONE LABORATORIES

World center of communications research and development

Journal
of
The Franklin Institute

MANAGING EDITOR, NANCY S. GLENN

Editor Emeritus, HENRY B. ALLEN

Associate Editors:

CHARLES B. BAZZONI
WILLIAM H. BOGHOSIAN
CHARLES L. CRITCHFIELD
RUPEN EKSERGIAN
VIRGIL MORING FAIRES
PAUL D. FOOTE
WALTER C. JOHNSON
ROBERT S. LIVINGSTON
C. E. K. MEES
F. J. MURRAY
W. F. G. SWANN
HUGH S. TAYLOR
MERLE A. TUVE

ARTHUR L. DAY, *Emeritus*

Committee on Publications:

HENRY B. ALLEN
LIONEL F. LEVY, *Chairman*
CLARENCE L. JORDAN
PHILIP H. WARD, JR.

Vol. 269 **JUNE, 1960** **No. 6**

CONTENTS

The Thunder-Storm as a Chemical Phenomenon ..... R. v. KONOW 439

On Helmholtz's Theorem in Multiply-Bounded and Multiply-Connected Regions ..... J. VAN BLADEL 445

Stresses in Certain Thin Plates Rotating about an Axis Lying in Their Middle Plane ..... R. S. DHALIWAL 463

Some Considerations of the Mechanics and Design Limitations of Autofrettage ..... J. H. FAUPEL 474

Minutes of the Stated Meeting of The Franklin Institute, May 18, 1960 490

Book Notes 491

Publications Received 494

National Bureau of Standards News 495

Current Topics 498

Index to Volume 269 506

Published monthly by The Franklin Institute of the State of Pennsylvania, Benjamin Franklin Parkway at Twentieth St., Philadelphia 3, Pennsylvania. Printed by Lancaster Press, Inc., Prince and Lemon Sts., Lancaster, Pennsylvania.

Subscription rates: Domestic, $10.00 a year; Foreign, $11.00 a year. Single current issues, $1.00. Reprints of papers published prior to July, 1959, 50 cents per installment. Publication of reprints for sale was discontinued in July, 1959. All inquiries regarding subscriptions, change of address, sale of back issues and reprints, should be addressed to the Philadelphia office.

Indexes to the semi-annual volumes of the JOURNAL are published in the June and December numbers.

Copyright © 1960, by The Franklin Institute of the State of Pennsylvania.

THE FRANKLIN INSTITUTE
LABORATORIES FOR RESEARCH AND DEVELOPMENT

announce opportunities for:

ELECTRICAL ENGINEERS

SYSTEMS ENGINEERS

ORGANIC CHEMISTS

OPERATIONS RESEARCH SPECIALISTS

MECHANICAL ENGINEERS

EXPERIMENTAL PSYCHOLOGISTS

METALLURGISTS

INDUSTRIAL ENGINEERS

PHYSICISTS

APPLIED MATHEMATICIANS

to work on challenging problems in research

Send complete resume to:

Mr. John E. Christ, *Director of Personnel*
THE FRANKLIN INSTITUTE
Philadelphia 3, Pa.

Journal
of
The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 269 JUNE, 1960 No. 6

THE THUNDER-STORM AS A CHEMICAL PHENOMENON

BY

R. v. KONOW [1]

ABSTRACT

Notwithstanding the fact that many valuable scientific results on electric charges in the atmosphere have been published during the last twenty years, the main question, the origin of thunder-storms, is yet unsolved. This difficult problem is generally assumed to be purely in the field of physics; however, there are many circumstances indicating chemical reactions, which we may take into consideration. For example, chemical changes of oxygen and water molecules in the air may promote the storage of electricity which discharges through lightning. In this paper, the author discusses some chemical reactions which may be responsible for the accumulation of electricity.

1. OXYGEN MOLECULES IN THE AIR DISSOCIATE INTO OXYGEN ATOMS

Dry oxygen is very stable at low temperatures, but in connection with moisture its activity is increased. This we observe in all oxidizing reactions, as well in organic as in inorganic chemistry. In 1952 the author proposed (1)[2] that the essential condition for oxidation is the association of the reacting molecules through *hydrogen bonds* over which electrons pass from the reducing molecule to the oxidizing one.

The reaction between dry substances seems to be hindered by the outer shells of electrons which cover all molecules causing them to repel each other; but, if protons are present building hydrogen bridges, the distances between the molecules will be diminished facilitating the redox-reaction.

This assumption for oxidizing reactions at low temperatures we see accepted also by other scientists (2). On the other hand, at high temperatures the thermal collisions between molecules are strong enough to overcome the repulsion effect. Thus electrons may jump without hydrogen bridges which characterize the ignition temperature.

[1] Helsingfors, Finland.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

High in the atmosphere the temperature is low. Therefore the direct dissociation of oxygen molecules in atoms is not very likely. On the other hand, if moisture is present, oxygen combines with water molecules through hydrogen bonds in the way shown in Fig. 1. This process may be described as follows: An oxygen molecule condenses with two water molecules through hydrogen bridges. In the high atmosphere this complex absorbs light energy which elevates electrons to higher energy levels. In this excited state two electrons pass over the hydrogen bridges from the water molecules to the oxygen forming two molecules of hydrogen peroxide. These separate after the electron transfer is concluded. As an unstable substance hydrogen peroxide soon decomposes in a water molecule and an *oxygen atom*.

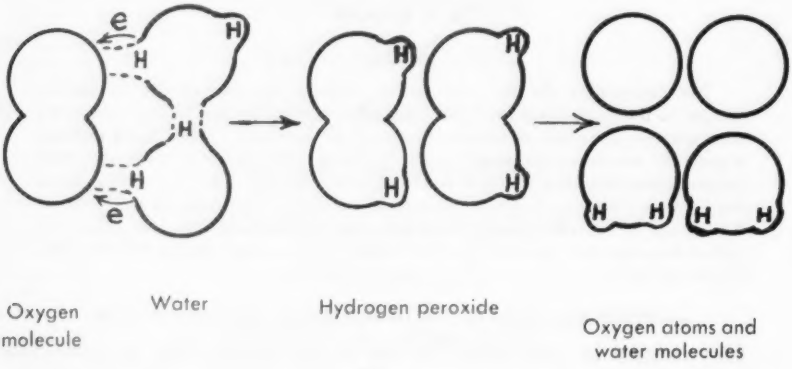


FIG. 1.

Thus water reacts as a catalyst dissociating oxygen into oxygen atoms. The catalytic role of water in the atmosphere is important. At first this mechanism needs low temperatures for the condensation, which means that the reaction occurs high in the atmosphere, but not near the Earth's surface. Secondly, oxygen molecules are very stable. Their splitting seems to need the ultraviolet radiation which is intense enough only in the higher atmosphere.

2. ORIGIN OF THE POSITIVE CHARGE IN THE ATMOSPHERE

According to many scientists the main part of the atmosphere is positively charged. Wilson and Simpson (3) illustrate this through Fig. 2.

Israël (4) points out that the whole atmosphere reacts in the same way as an electric battery in which the thunder-storms act as the generator and the electrically calm part of the whole atmosphere as the conducting wire.

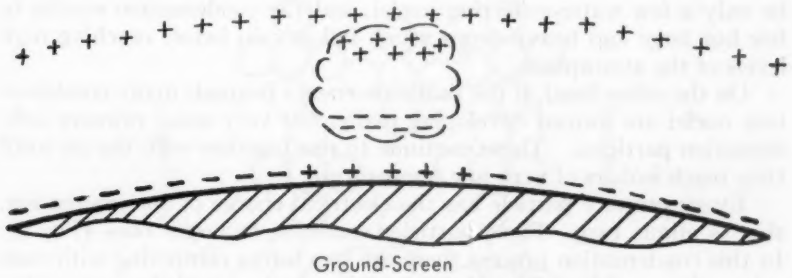


FIG. 2. Atmospheric electricity according to Wilson.

The author assumes that this phenomenon has the following origin (shown in Fig. 3). Oxygen atoms have only six electrons in their outer shell. This is a very unstable structure which has a tendency to absorb free electrons from the atmosphere forming the stable electron octet. Thus oxygen ions, O^{--}, are formed and the air ionized. Because the movements in the upper part of the atmosphere are very rapid, this effect expands over the whole globe as a deficiency of electrons compared with the charge of the Earth itself. This indicates the positive charge of the atmosphere, since the oxygen ions leave the atmosphere with the rains, as condensing nuclei.

3. DEVELOPMENT OF THUNDER-CLOUDS

Convection currents are caused by the energy of the sun's radiation. However, the heat energy is soon consumed by the adiabatic work of the rising air. On the other hand, if the air is moist, heat energy is liberated during the course upwards because of condensation of the water content. Thus the highest convection currents develop in those districts where both radiation and evaporation are intense. There the majority of thunder-storms occur.

If radiation is smooth, only a few oxygen molecules will be split into atoms, and thus also only a few oxygen ions formed. There will

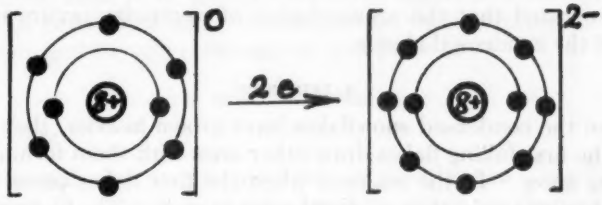
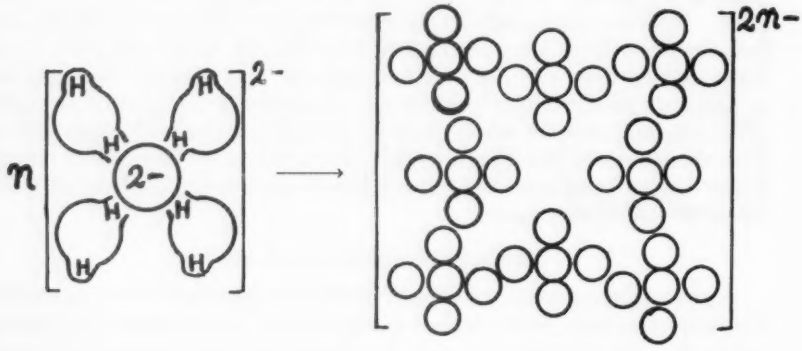


FIG. 3.

be only a few water-collecting nuclei, and the condensation results in few but large and heavy drops which fall as rain before reaching high levels of the atmosphere.

On the other hand, if the radiation energy is great, many condensation nuclei are formed developing many, but very small primary condensation particles. These continue to rise together with the air until they reach isobars of very low temperature.

Every primary particle has the electrical charge of one oxygen ion, that is minus two. These particles condense to larger ones (Fig. 4). In this condensation process there are two forces competing with each other: the repulsing effect of the negative charges, and the force of the combining hydrogen bonds. These are weak in water but strong in ice, thus giving the possibility of electric storage.



n number of primary charges

A condensed charge

FIG. 4.

The necessity of cold and ice-formation for the development of thunder-clouds is thus in accordance with the observations of Norinder (5), who stated that the accumulation of electricity occurs in the ice phase of the condensed clouds.

4. LIGHTNING

When the condensed snowflakes have grown heavier, they begin to fall. The first falling flakes draw other ones with them forming a cone of falling snow. In the moment when the first flakes come into contact with warmer air, their surface begins to melt. The hydrogen bonds are thus weakened, which releases the charges as an explosion causing the whole cone, and even the whole cloud, to discharge at the same moment.

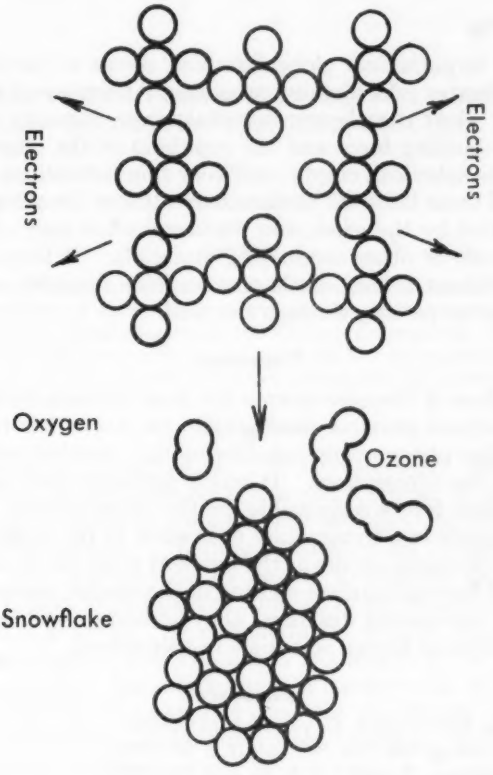


FIG. 5.

5. FORMATION OF OZONE

Snowflakes which are already melting may be transformed into oxygen ions and water (Fig. 5). On the other hand, the flakes which are in a frozen state have their atoms firmly together. This promotes formation of covalent bonds. Consequently, in this condition formation of ozone and oxygen molecules is very probable.

6. OTHER PHENOMENA

St. Elmo's Fire

Other related phenomena could also be explained in a similar way. For instance, St. Elmo's Fire may have its origin in thunder-clouds which didn't develop until lightning, but are carried by the wind to cooler districts, where the snowflakes without melting may reach the earth's surface where they discharge.

Globe Lightning

Even the hypothetical globe lightning seems to have some possibility of existence. As already mentioned, there are two competing forces acting when the electric snowflakes are formed: the hydrogen bonds as the binding force and the repulsion of the charges as a hindrance. Thus spherical, empty, and very thin formations of snow seem probable. If these balls fall to the earth without breaking, they would be easily carried by the wind, and discharge when they come into contact with metals or other conducting materials. If they touch a wall, they may, without a loud discharge, combine invisibly with it giving the impression of passing through the wall.

SUMMARY

The problem of thunder-storms has been investigated mainly from meteorological and physical viewpoints. In this article the author discusses the same phenomenon considering the chemical reactions of the molecules in the atmosphere. It seems probable that *hydrogen bonds* are the essential forces responsible for the accumulation of electricity, which occurs only in a frozen state or in snow in the upper atmosphere, and that the discharge occurs at the moment when the hydrogen bridges are weakened through surface melting of the electric snowflakes. Thus the chemical viewpoints discussed above conform with experimentally measured results of known scientists in this sphere.

REFERENCES

(1) R. v. KONOW, *TAPPI* (U. S. A), Vol. 35, p. 64A (1952).
(2) R. CRIEGEE, *Liebigs Ann.*, Vol. 599, Part II, p. 101 (1956).
(3) GEORGE C. SIMPSON, *Weather*, Vol. 4, No. 3, p. 134 (1949).
(4) H. ISRAËL, *Arch. Meteor. Geophys. and Bioklim.*, Vol. (A) 3, pp. 1–16 (1950).
(5) H. NORINDER, "Experimental Lightning Research," JOUR. FRANKLIN INST., Vol. 253, pp. 471–504 (1952); "Some Aspects and Recent Results of Electromagnetic Effects of Thunderstorms," *ibid.*, Vol. 244, pp. 109–130 and 167–207 (1947); *Tellus*, Vol. 1, No. 2, p. 1 (1949); *Arkiv. Geofys.*, Vol. 2, No. 6, p. 533 (1958).
(6) E. L. HILL, "Electromagnetic Radiation from Lightning Strokes," JOUR. FRANKLIN INST., Vol. 263, p. 107 (1957).
(7) B. F. J., SCHONLAND, "Atmospheric Electricity," New York, McGraw-Hill Book Co., Inc., 1950.

ON HELMHOLTZ'S THEOREM IN MULTIPLY-BOUNDED AND MULTIPLY-CONNECTED REGIONS*

BY

J. VAN BLADEL [1]

ABSTRACT

The splitting of a continuous vector function into a gradient plus a curl is possible in an infinite number of ways. The present paper is basically concerned with two particular uniquely determined splittings called electric and magnetic. Eigenvector expansions suitable for these splittings are examined, and the application of these expansions to the calculation of the various terms of the splitting is considered.

I. INTRODUCTION

A certain number of papers (1, 2, 3)[2] have recently been published on the possibility of representing a solenoidal vector by the curl of a vector potential in multiply-bounded and multiply-connected regions, and on the related problem of finding a complete set of eigenvectors for such regions. The present work endeavors to examine these problems in the light of Helmholtz's theorem, that is, of the splitting of a vector into a curl and a gradient. It culminates in a table which, notwithstanding its seemingly trivial character, is thought to be of potential interest to physicists. The mathematical background to the present study was provided by Weyl shortly after the turn of the century.

Let $\bar{a}$ be a piecewise continuous vector function. It is sought to investigate whether, and how, $\bar{a}$ can be split into

$$\bar{a} = \text{grad } \Omega + \text{curl } \bar{w} \tag{1}$$

in a finite regular region V, which can be either simply-bounded and simply-connected as in Fig. 1*a*, simply-connected and multiply-bounded as in Fig. 1*b* (and for the sake of clarity doubly-bounded regions only will be considered), or simply-bounded and multiply-connected as in Fig. 1*c*. There again doubly-connected regions, of the type encountered in toroidal particle accelerators, will be examined at the exclusion of more complicated structures. Generalizations are obvious. The gradient and curl terms of the splitting will be called "longitudinal" and "transverse" terms respectively.

* AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AT(11-1)-384.

[1] Midwestern Universities Research Association and Department of Electrical Engineering, University of Wisconsin, Madison, Wisc.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

First of all, it is apparent that the problem can be solved in at least one way, for the equation

$$\nabla^2\bar{u} = \text{grad div } \bar{u} - \text{curl curl } \bar{u} = \bar{a} \tag{2}$$

admits at least one solution in the regions considered above. As a matter of fact, it admits an infinity of solutions, and one suspects right away that the splitting must be possible in an infinite number of fashions. This is confirmed by the remark that vectors do exist which can be written either as a gradient or a curl. Addition of such a vector to the longitudinal term, and subtraction from the transverse term, yields a new splitting. It will consequently be possible to impose certain conditions on Ω or $\bar{w}$ (for example that $\bar{w}$ be perpendicular to the boundary surface) to restrict the generality of the splitting, and make it unambiguous. The "electric," "magnetic," and "mixed" splittings, to be discussed hereafter, result from well-chosen additional conditions and are of particular interest for solenoidal and irrotational vectors. Their use is fruitful in the analysis of electromagnetic fields in finite regions. Other splittings (4), not to be considered here, are of interest for other branches of physics.

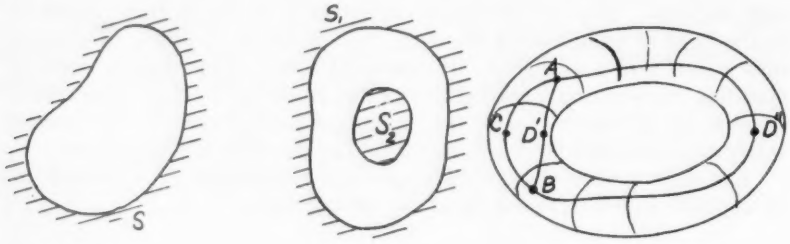


FIG. 1. Typical columns.

II. THE ELECTRIC SPLITTING

A. Preliminary Considerations on Irrotational Vectors (curl $\bar{a} = 0$)

An application of Stokes' theorem shows that, in Regions I and II, $\int_A^B \bar{a}\cdot\overline{dl}$ is independent of the path joining A to B provided this path is entirely contained in the region throughout which $\bar{a}$ is irrotational. It is consequently possible to express $\bar{a}$ as grad Ω, where Ω is defined by the relation $\Omega(B) = \Omega(A) + \int_A^B \bar{a}\cdot\overline{dl}$. Things are somewhat different in Region III. The contour integral of $\bar{a}$ still vanishes along a curve such as $ACBD'$, that does not "encircle" the hole (that is, that can be

reduced to a point without leaving Region III), but takes, in general, a non-zero value T for a contour such as $ACBD''$ that encircles the hole once. This value T, called the period, is independent of the encircling contour. There now exists a multivalued function Ω such that $\bar{a} = \text{grad } \Omega$. The values of this function at any given point differ by an integral number of times T.

B. Existence and Uniqueness

The electric splitting is generated by the solution of the problem

$$\nabla^2 \bar{f} = \text{grad div } \bar{f} - \text{curl curl } \bar{f} = \bar{a} \qquad \text{with} \qquad \left.\begin{matrix} \text{div } \bar{f} = 0 \\ \bar{u}_n \times \bar{f} = 0 \end{matrix}\right\} \text{ on } S \quad (3)$$

where $\bar{u}_n$ is the unit vector directed along the outside normal to S. The *uniqueness* of the solution should be investigated first. If there were two different solutions $\bar{f}_1$ and $\bar{f}_2$, their difference $\bar{f}'$ would satisfy $\nabla^2 \bar{f}' = 0$. Application of the formula

$$\iiint_V [\bar{v} \cdot \nabla^2 \bar{w} + \text{div } \bar{v} \cdot \text{div } \bar{w} + \text{curl } \bar{v} \cdot \text{curl } \bar{w}] dV = \iint_S [(\bar{u}_n \cdot \bar{v}) \text{ div } \bar{w} + (\bar{v} \times \text{curl } \bar{w}) \cdot \bar{u}_n] dS \quad (4)$$

to $\bar{v} = \bar{w} = \bar{f}'$ shows that $\text{div } \bar{f}' = \text{curl } \bar{f}' = 0$ throughout the region. Such a vector will be called "source-free." It can be represented as grad φ, where φ is a harmonic function which, because of the perpendicularity of $\bar{f}'$ to S, takes a constant value along each of the closed surfaces of which the boundary consists. In simply-connected regions, φ is obviously single-valued. In Region III, φ is single-valued too, because the period T is equal to zero. This is easily shown by particularizing the contour over which $\int \bar{f}' \cdot \overline{dl}$ is evaluated to lie on the surface S, where $\bar{f}'$ is perpendicular to $\overline{dl}$. It is now possible to show that $\bar{f}'$ must be zero in Regions I and III, by applying the relation

$$\iiint_V [A \nabla^2 B + \text{grad } A \cdot \text{grad } B] dV = \iint_S A \frac{\partial B}{\partial n} dS \quad (5)$$

valid for single-valued functions, to $A = B = \varphi - \varphi_S$, where φ_S is the constant value which φ takes on the boundary surface. In Region II however, there are non-zero solutions for $\bar{f}'$, all multiples of each other, and derived from potentials φ which take different constant values on S_1 and S_2. Let $\bar{f}_0 = \text{grad } \varphi_0$ be the "normalized" solution, determined by the condition that $\iiint_V |\bar{f}_0|^2 dV = 1$. The following statement

can now be made: If $\bar{f}$ exists, it is unique, except in Region II where it is determined with the exception of a multiple of $\bar{f}_0$. In the latter region, the various determinations lead to identical longitudinal and transverse terms, because $\operatorname{div} \bar{f}_0 = 0$ and $\operatorname{curl} \bar{f}_0 = 0$. One concludes that there is at most one way to split a vector electrically.

For a proof of the *existence* of a solution, one has to turn to a paper by Weyl (5), where the existence of a Green's dyadic $\mathcal{G}_e(\bar{r}, \bar{r}')$, such that

$$\bar{f}(\bar{r}) = \iiint_V \mathcal{G}_e(\bar{r}, \bar{r}') \cdot \bar{a}(\bar{r}') dV' \tag{6}$$

is established for a simply-bounded region. This relation implies that the Laplacian of the second member, taken with respect to the $\bar{r}$ coordinates, reproduces $\bar{a}(\bar{r})$. In other words, the solution of Eq. 6, considered as an integral equation in $\bar{a}$ with given second member $\bar{f}$, is $\bar{a} = \nabla^2 \bar{f}$, provided $\bar{f}$ does not have any tangential components on S, and possesses a piecewise continuous Laplacian. In particular, the homogeneous integral equation derived from Eq. 6 by setting the second member $\bar{f}$ equal to zero, has $\bar{a} = 0$ as its only solution. This important property will be used later in the argument.

In the course of proving the existence of $\mathcal{G}_e$, Weyl establishes the following properties of the dyadic:

(*a*) it is symmetrical, that is, $\mathcal{G}_{ik}(\bar{r}, \bar{r}') = \mathcal{G}_{ki}(\bar{r}', \bar{r})$

(*b*) it behaves, for $\bar{r} \to \bar{r}'$, in such a manner that

$$\lim_{\bar{r} \to \bar{r}'} \mathcal{G}_{ik}(\bar{r}, \bar{r}') \leq \frac{\text{constant}}{|\bar{r} - \bar{r}'|}.$$

When the region is multiply-bounded, no unique solution $\bar{f}$ exists and, consequently, no Green's dyadic in the sense of Eq. 6. There is a way out of this difficulty, however, if one decides to split $\bar{f}$ and $\bar{a}$ into two components:

$$\bar{f} = \bar{f}_0 \iiint_V \bar{f} \cdot \bar{f}_0 dV + \bar{f}_c$$

$$\bar{a} = \bar{f}_0 \iiint_V \bar{a} \cdot \bar{f}_0 dV + \bar{a}_c$$

All the possible solutions $\bar{f}$ have the same "core" part $\bar{f}_c$, which is, consequently, uniquely determined. The two parts of the splitting can be thought of as the functional projections of $\bar{f}$ (and $\bar{a}$) on $\bar{f}_0$ and on the subspace $\bar{f}_c$ orthogonal to $\bar{f}_0$, respectively. Notice that both projections are orthogonal to each other, in the sense that $\iiint \bar{f}_c \cdot \bar{f}_0 dV$. The problem

$$\nabla^2 \bar{f}_c = \bar{a}_c \tag{7}$$

can be proved to have a unique solution, which Weyl shows to be obtainable in the form

$$\bar{f}_e(\bar{r}) = \iiint \mathcal{G}_e'(\bar{r}, \bar{r}') \cdot \bar{a}_e(\bar{r}') dV' \tag{8}$$

where $\mathcal{G}_e'$ is an electric Green's dyadic in the extended sense. This dyadic has the same symmetry property, and the same behavior for $\bar{r} \to \bar{r}'$ as the usual Green's dyadic $\mathcal{G}_e$. It now appears that the electric splitting is always possible in the form

$$\bar{a} = \bar{f}_0 \iiint_V \bar{a} \cdot \bar{f}_0 dV + \text{grad div } \bar{f} - \text{curl curl } \bar{f} \tag{9}$$

where the term in $\bar{f}_0$ exists in Region II only, and where the three parts of the splitting are uniquely determined. The name "longitudinal" will henceforth be reserved for the second term grad div $\bar{f}$.

C. Further Properties. Application to Solenoidal and Irrotational Vectors

1. The longitudinal term can be obtained by solving a potential problem. Taking the divergence of both terms of Eq. 9 yields, indeed:[3]

$$\nabla^2(\text{div } \bar{f}) = \text{div } \bar{a} \qquad \text{with} \qquad \text{div } \bar{f} = 0 \qquad \text{on } S.$$

The longitudinal term turns out to be the electric field created in region V by a charge density ϵ_0 div $\bar{a}$, the walls being metallized and grounded.

2. The three terms of the splitting are functionally orthogonal to each other. The orthogonality of $\bar{f}_0$ and grad div $\bar{f}$ can be established by applying the equation

$$\iiint_V \bar{b} \cdot \text{grad } \varphi dV = \iint_S \varphi(\bar{u}_n \cdot \bar{b}) dS - \iiint_V \varphi \text{ div } \bar{b} dV \tag{10}$$

to $\bar{b} = \bar{f}_0$, $\varphi = \text{div } \bar{v}$. The orthogonality of $\bar{f}_0$ and grad div $\bar{f}$ to $-$curl curl $\bar{f}$ results from applying

$$\iiint_V \bar{b} \cdot \text{curl } \bar{a} dV = \iint_S \bar{u}_n \cdot (\bar{a} \times \bar{b}) dS + \iiint_V \bar{a} \cdot \text{curl } \bar{b} dV \tag{11}$$

to $\bar{a} = -$ curl $\bar{f}$ and to $\bar{b} = \bar{f}_0$ and grad div $\bar{f}$ in succession.

3. When $\bar{a}$ is solenoidal, that is, div $\bar{a} = 0$, the longitudinal term vanishes and the splitting takes the form

$$\bar{a} = \bar{f}_0 \iiint \bar{a} \cdot \bar{f}_0 dV - \text{curl curl } \bar{f}. \tag{12}$$

[3] It is assumed that $\bar{a}$ has piecewise continuous first derivatives.

The solenoidal vector $\bar{a}$ cannot, in general, be derived from a vector potential in a multiply-bounded region of Type II. To justify this assertion, consider first the fact that $\bar{f}_0$ cannot be derived from a vector potential. If, indeed, $\bar{f}_0$ were equal to curl $\bar{M}$, an application of the formula

$$\iiint_V [\text{curl } \bar{a} \cdot \text{curl } \bar{b} - \bar{a} \cdot \text{curl curl } \bar{b}] dV = \iint_{S_1+S_2} \bar{u}_n \cdot (\bar{a} \times \text{curl } \bar{b}) dS \quad (13)$$

to $\bar{a} = \bar{b} = \bar{M}$ would yield $\iiint_V |\text{curl } \bar{M}|^2 dV = 0$, that is, $\bar{f}_0$ equal to zero, an obvious contradiction. Clearly then, $\bar{a}$ derives from a vector potential, if, and only if, its functional projection on $\bar{f}_0$ is zero, that is, $\iiint_V \bar{a} \cdot \bar{f}_0 dV$ vanishes. But this implies that

$$\begin{aligned} \iiint_V \bar{f}_0 \cdot \bar{a} dV &= \iiint_V \text{grad } \varphi_0 \cdot \bar{a} dV \\ &= \iint_{S_1+S_2} \varphi_0 (\bar{a} \cdot \bar{u}_n) dS - \iiint_V \varphi_0 \text{ div } \bar{a} dV \\ &= \varphi_{01} \iint_{S_1} (\bar{a} \cdot \bar{u}_{n1}) dS_1 + \varphi_{02} \iint_{S_2} (\bar{a} \cdot \bar{u}_{n2}) dS_2 = 0 \end{aligned} \quad (14)$$

where φ_{01} and φ_{02} are the two (unequal) values of φ_0 on S_1 and S_2. On the other hand, an application of Gauss' theorem to the solenoidal vector $\bar{a}$ yields

$$\iiint_V \text{div } \bar{a} dV = 0 = \iint_{S_1} (\bar{a}, \bar{u}_{n1}) dS_1 + \iint_{S_2} (\bar{a} \cdot \bar{u}_{n2}) dS_2. \quad (15)$$

A comparison of Eqs. 14 and 15 shows that a solenoidal vector in a doubly bounded region derives from a vector potential if, and only if, its flux through *each* of the two boundary surfaces is equal to zero. This result has previously been obtained by Stevenson (2) by somewhat different methods.

4. When $\bar{a}$ is irrotational, and possesses a tangential component along S, all terms of the electric splitting are present. Both $\bar{f}_0$ and the longitudinal part being perpendicular to S, it is the transverse term which will be responsible for the tangential component. The main function of the transverse term is, so to speak, to account for that component, because $-$ curl curl $\bar{f}$ vanishes when $\bar{a}$ is perpendicular to S. This can be proved by applying (13) to $\bar{a} = \bar{b} = \text{curl } \bar{f}$

$$\iiint_V [\text{curl curl } \bar{f}]^2 dV = \iiint_V \text{curl } \bar{f} \cdot \text{curl curl curl } \bar{f}\, dV + \iint_S \bar{u}_n \cdot [\text{curl } \bar{f} \times \text{curl curl } \bar{f}]\, dS.$$

The second member vanishes because curl curl curl $\bar{f} = -$ curl $\bar{a}$ is equal to zero, by hypothesis, and because curl curl $\bar{f}$ is perpendicular to S. This implies the vanishing of the first member, and of curl curl $\bar{f}$ in consequence.

The irrotational character of $\bar{a}$ allows representation of that vector as the gradient of a scalar potential. A look at Eq. 9 shows that the transverse term can, consequently, also be derived from a scalar potential and, furthermore, one that is harmonic throughout V. A consequence of this fact, which rests on properties to be enunciated later, is that the transverse term cannot be purely tangential in a simply-connected region. It then contributes part of the normal component of $\bar{a}$ at the surface, the other part being furnished by the $\bar{f}_0$ and longitudinal terms.

5. The electric splitting can be put in the form

$$\bar{a} = \bar{f}_0 \iiint \bar{a} \cdot \bar{f}_0 dV + \text{grad } \varphi + \text{curl } \bar{w} \tag{16}$$

where φ vanishes on the boundary. Conversely, if a splitting such as (16) has been achieved, it must be the electric splitting. Taking the divergence of both members shows, indeed, that $\nabla^2 \varphi = \text{div } \bar{a}$. This equation, together with the condition for φ at the boundary, is also satisfied by div $\bar{f}$. The solution of this potential problem being unique, φ *must* be div $\bar{f}$, and the splitting (16) is unique. Notice that $\bar{w}$ is a tangential vector, being the curl of $\bar{f}$, a vector perpendicular to the boundary.

D. Eigenvector Expansions Suitable for the Electric Splitting

In the search for a complete set of eigenvectors, some fundamental properties of integral operators will first be reviewed (6). Let $G(x, x')$ be a symmetrical kernel (that is, such that $G(x, x') = G(x', x)$), and let G be square integrable in the plane domain $a \leq x \leq b$, $a \leq x' \leq b$. Then there exists a sequence of orthogonal eigenfunctions $\varphi_m(x)$, and corresponding non-zero eigenvalues μ_m, such that

$$\int_a^b G(x, x') \varphi_m(x') dx' + \mu_m \varphi_m(x) = 0$$

where the φ_m belong to the space of square integrable functions. Any square integrable function ψ in (a, b) admits the expansion

$$\psi(x) = \sum_m \varphi_m(x) \int_a^b \varphi_m(x')\psi(x')dx' + \theta(x)$$

where the convergence is in the mean, and the rest term $\theta(x)$ is a solution of

$$\int_a^b G(x, x')\theta(x')dx' = 0.$$

When the latter integral equation admits zero as its only solution, no rest term is necessary, and the set of the φ_m is complete.

The same sort of expansion theorem can be established (**7, 8**) in three-dimensional space, where the elements φ_m and ψ are replaced by square integrable three-dimensional vectors, and $G(x, x')$ is replaced by a symmetric dyadic $\mathcal{G}$ (that is, such that $\mathcal{G}_{xy}(\bar{r}, \bar{r}') = \mathcal{G}_{yx}(\bar{r}', \bar{r})$) having the property that $\iiint_V [\bar{a}(\bar{r}') \cdot \mathcal{G}(\bar{r}, \bar{r}')]^2 dV'$ is a finite and continuous function of $\bar{r}$ whenever $\bar{a}$ is continuous. The two dyadics $\mathcal{G}_e$ and $\mathcal{G}_e'$ have sufficiently mild discontinuities at $\bar{r} = \bar{r}'$ to satisfy these requirements. Consequently, in Regions I and III, the eigenvectors $\bar{F}_m$ of

$$\iiint_V \mathcal{G}_e(\bar{r}, \bar{r}') \cdot \bar{F}_m(\bar{r}')dV' + \mu_m \bar{F}_m(\bar{r}) = 0 \qquad (17)$$

form a complete set. The homogeneous integral equation (6) has indeed been shown to have zero as its only solution. It will be noticed, by taking the Laplacian of both sides, that

$$\bar{F}_m(\bar{r}) + \mu_m \nabla^2 \bar{F}_m(\bar{r}) = 0.$$

On the other hand, the nature of $\mathcal{G}_e$ is such that the vector represented by the integral in Eq. 17 satisfies $\bar{u}_n \times \bar{F}_m = \text{div } \bar{F}_m = 0$ on S. The $\bar{F}_m$ are consequently also the eigenvectors of the differential operator ∇^2 corresponding to these boundary conditions, the eigenvalues being now $\lambda_m = \dfrac{1}{\mu_m}$. It is shown in Appendix I that these eigenvectors fall into two categories:

(*a*) solenoidal eigenvectors $\bar{e}_m$, solutions of

$$-\text{curl curl } \bar{e}_m + \lambda_m' \bar{e}_m = 0 \qquad \bar{u}_n \times \bar{e}_m = 0 \qquad \text{on } S \qquad (18)$$

(*b*) irrotational eigenvectors $\bar{f}_m = \text{grad } \varphi_m$, obtained from the eigenfunctions φ_m of

$$\nabla^2 \varphi_m + \lambda_m'' \varphi_m = 0. \qquad \varphi_m = 0 \text{ on } S \qquad (19)$$

These eigenvectors are all orthogonal to each other, and it will be assumed that they have been normalized.

In Region II, similarly, the eigenvectors $\bar{F}_m'$ of $\iiint_V \mathcal{G}_e'(\bar{r}, \bar{r}')\, dV'$ form a complete set with respect to the subspace of vectors that are orthogonal to $\bar{f}_0$. This restriction is made necessary by the fact that $\nabla^2 \iiint_V \mathcal{G}_e'(\bar{r}, \bar{r}') \cdot \bar{a}(\bar{r}')\, dV'$ is equal to $\bar{a}_c(\bar{r})$. In consequence, if the integral $\iiint_V \mathcal{G}_e'(\bar{r}, \bar{r}') \cdot \bar{a}(\bar{r}')\, dV'$ vanishes for all $\bar{r}$, we can conclude that the "core" part $\bar{a}_c$ of $\bar{a}$ vanishes, but not that $\bar{a}$ itself is equal to zero. In fact, any multiple of $\bar{f}_0$ is a solution of the homogeneous integral equation. It then follows that the "core" part of any square integrable vector can be expanded in the $\bar{F}_m'$. In formula:

$$\bar{a}(\bar{r}) = \bar{f}_0(\bar{r}) \iiint_V \bar{a}(\bar{r}') \cdot \bar{f}_0(\bar{r}')\, dV' + \sum_m \bar{F}_m'(\bar{r}) \iiint_V \bar{F}_m'(\bar{r}') \cdot \bar{a}(\bar{r}')\, dV'. \quad (20)$$

It is found, by arguments similar to those used for $\bar{F}_m$, that the $\bar{F}_m'$ belong to the subspace of vectors orthogonal to $\bar{f}_0$, and that they are the eigenvectors of ∇^2 belonging to eigenvalues different from zero. It is a simple matter, indeed, to establish the orthogonality of the latter eigenvectors to $\bar{f}_0$ by applying the formula

$$\iiint_V [\bar{a} \cdot \nabla^2 \bar{b} - \bar{b} \cdot \nabla^2 \bar{a}]\, dV = \iint [(\bar{a} \cdot \bar{u}_n) \operatorname{div} \bar{b} - (\bar{b} \cdot \bar{u}_n) \operatorname{div} \bar{a} + (\bar{u}_n \times \bar{a}) \cdot \operatorname{curl} \bar{b} - (\bar{u}_n \times \bar{b}) \cdot \operatorname{curl} \bar{a}]\, dS \quad (21)$$

to $\bar{a} = \bar{f}_0$, $\bar{b} = \bar{F}_m'$. It is apparent from Eq. 20, by the way, that $\bar{f}_0$ can be included in the set of eigenvectors to make it complete with respect to the space of square integrable vectors. The $\bar{F}_m'$ can also be subdivided into categories (18) and (19) already introduced for simply bounded regions. It is now possible to expand each term of the electric splitting (16) in the $\bar{f}_0$, the $\bar{e}_m$ and $\bar{f}_m$. The coefficients of the expansion can be found by repeated use of formulas 10, 11, and 13. The resulting formula,[4] of great importance for practical applications, is:

$$\bar{a}(\bar{r}) = \underbrace{\bar{f}_0 \left[\iint_{S_1+S_2} (\bar{u}_n \cdot \bar{a})\, \varphi_0 dS - \iiint_V \varphi_0 \operatorname{div} \bar{a}\, dV \right]}_{\text{Term present in Region II only}}$$

(*Continued on next page*)

[4] This formula is valid only when the first order partial derivatives of $\bar{a}$ are sufficiently regular for Gauss' and Stokes' theorems to be applicable.

$$+\underbrace{\sum_m \bar{f}_m\left[-\iiint \varphi_m \operatorname{div} \bar{a}\, dV\right]}_{\text{grad div } \bar{f}}$$

$$+\underbrace{\sum_m \frac{\bar{e}_m}{\lambda_m'}\left[\iiint_V \operatorname{curl} \bar{e}_m \cdot \operatorname{curl} \bar{a}\, dV - \iint_S (\bar{u}_n \times \bar{a}) \cdot \operatorname{curl} \bar{e}_m dS\right]}_{-\operatorname{curl}\operatorname{curl} \bar{f}} \quad (22)$$

All the results relative to solenoidal and irrotational vectors can be obtained directly from this formula.

A short additional remark will be of interest here. Formula 22 shows that, in a region of Type I, a sourceless vector is expandable in the solenoidal vectors $\bar{e}_m$, but does not have any functional projection on the $\bar{f}_m = \operatorname{grad} \varphi_m$. On the other hand, a sourceless vector can certainly be represented as the gradient of a scalar potential φ. The latter can be expanded in the complete set φ_m, and one obtains a representation of $\bar{a}$ as $\operatorname{grad}[c_1\varphi_1 + \cdots + c_n\varphi_n + \cdots]$. Here then is an interesting example of a situation where the gradient of a sum is not the sum of the gradients, that is, $\bar{a}$ is not $c_1 \operatorname{grad} \varphi_1 + \cdots + c_n \operatorname{grad} \varphi_n + \cdots$ throughout the region, a fact which is due to the non-continuous character of differential operators, and which parallels the well-known property that a Fourier series cannot, in general, be differentiated term by term.

III. THE MAGNETIC SPLITTING

A. Existence and Uniqueness

The magnetic splitting is generated by the solution of the problem

$$\nabla^2 \bar{h} = \operatorname{grad}\operatorname{div} \bar{h} - \operatorname{curl}\operatorname{curl} \bar{h} = \bar{a} \qquad \text{with} \qquad \left.\begin{matrix} \bar{u}_n \cdot \bar{h} = 0 \\ \bar{u}_n \times \operatorname{curl} \bar{h} = 0 \end{matrix}\right\} \text{ on } S. \quad (23)$$

The investigation of the solution of this problem proceeds along lines which parallel those used for the electric splitting. The uniqueness is proved by examining the solutions of the homogeneous problem $\nabla^2 \bar{h}' = 0$. An application of Eq. 4 shows these solutions to be sourceless vectors. They can consequently be put in the form $\bar{h}' = \operatorname{grad} \psi$, with ψ harmonic, and $\bar{h}'$ tangent to S $\left(\text{that is, } \dfrac{\partial \psi}{\partial n} = 0 \text{ on } S\right)$. An application of Eq. 5 shows that no such vector can exist in Regions I and II. This implies that the only solution to $\nabla^2 \bar{h}' = 0$ is $\bar{h}' = 0$ or, in other words, that an eventual solution of Eq. 23 must be unique. In Region III, however, there is an infinite number of non-zero solutions ψ, all multiples of each other. In a region bounded by a surface of revolution, for instance,

ψ is a multiple of $\theta +$ constant, and $\bar{h}'$ is a multiple of $\frac{\bar{u}_\theta}{r}$. Let $\bar{h}_0 = \text{grad } \psi_0$ be the normalized solution. The following statement can now be made: if $\bar{h}$ exists, it is unique, except in Region III, where it is determined with the exception of a multiple of $\bar{h}_0$. In the latter region the various determinations lead to identical longitudinal and transverse terms. Notice that $\bar{h}_0$, which derives from a multivalued scalar potential ψ_0, can also be derived from a vector potential. An application of the electric splitting to $\bar{h}_0$ shows, indeed, that there is a vector $\bar{f}$ such that $\bar{h}_0 = -\text{curl curl } \bar{f}$, where $\bar{f}$ is perpendicular to S. In consequence, h_0 is the curl of a vector tangent to S.

The existence of a solution follows, again, from results obtained by Weyl (**4**, **5**, **9**), who indicates that, in a simply connected region, a dyadic exists such that

$$\bar{h}(\bar{r}) = \iiint_V \mathcal{G}_H(\bar{r}, \bar{r}') \cdot \bar{a}(\bar{r}')\, dV'. \tag{24}$$

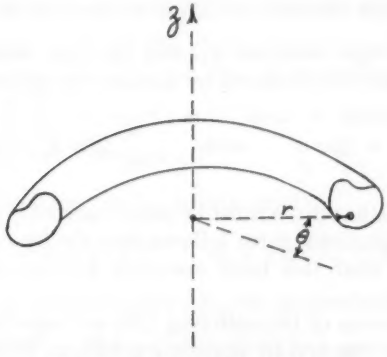


FIG. 2. A volume of revolution.

This dyadic has the symmetry property, and the behavior for $\bar{r} \to \bar{r}'$, which were associated with its electric counterpart. Also, the homogeneous integral equation possesses $\bar{a} = 0$ as its only solution. In a doubly-connected region, a magnetic Green's dyadic in the extended sense can be introduced, such that

$$\bar{h}_c(\bar{r}) = \iiint_V \mathcal{G}_H'(\bar{r}, \bar{r}') \cdot \bar{a}_c(\bar{r}')\, dV' \tag{25}$$

where $\bar{a}_c$ indicates the "core" part of a vector, obtained by subtracting from the vector its functional projection on $\bar{h}_0$. The dyadic $\mathcal{G}_H'$ has properties similar to those of $\mathcal{G}_e'$. It is now possible to state that: "the magnetic splitting is always possible in the form

$$\bar{a} = \bar{h}_0 \iiint_V \bar{a} \cdot \bar{h}_0 dV + \text{grad div } \bar{h} - \text{curl curl } \bar{h} \qquad (26)$$

where the term in $\bar{h}_0$ exists in Region III only, and where the three parts of the splitting are uniquely determined." The name "transverse" will from now on be reserved for the third term "$-$ curl curl $\bar{h}$."

B. Further Properties. Application to Solenoidal and Irrotational Vectors

1. The boundary conditions satisfied by $\bar{h}$ imply that $\dfrac{\partial}{\partial n}(\text{div } \bar{h}) = a_n$ on S. This result is easily obtained by projecting (23) on the normal to S. The vector curl $\bar{h}$ being perpendicular to S, its curl (that is, curl curl $\bar{h}$) is tangential to S (this property is established by applying Stokes' theorem to any small curve drawn on surface S, along which the circulation of curl $\bar{h}$ vanishes, so that the flux of curl curl $\bar{h}$ through *any* portion of S is zero, which implies that curl curl $\bar{h}$ is tangential). The projection of the left member of (23) is $\bar{u}_n \cdot \text{grad div } \bar{h} = \dfrac{\partial}{\partial n}(\text{div } \bar{h})$, the projection of the right member a_n, and the two must be equal. The scalar div $\bar{h}$ can now be obtained by solving the potential problem

$$\nabla^2 (\text{div } \bar{h}) = \text{div } \bar{a} \qquad \text{with} \qquad \frac{\partial}{\partial n}(\text{div } \bar{h}) = a_n \qquad \text{on } S \qquad (27)$$

and the restriction that div $\bar{h}$ must be single-valued. Once this problem is solved, the longitudinal term follows directly by taking the gradient of div $\bar{h}$. Notice that this term accounts for the normal component of $\bar{a}$ on S.

2. The three terms of the splitting (26) are functionally orthogonal. This can easily be checked by applying relations 10 and 11.

3. When $\bar{a}$ is solenoidal and tangential to S, the only single-valued solution to the potential problem (27) is div $\bar{h}$ = constant. The longitudinal term vanishes.

4. When $\bar{a}$ is irrotational, the transverse term $-$ curl curl $\bar{h}$ vanishes. This term can, indeed, be derived from a scalar potential, all other terms of Eq. 26 being the gradient of something. So, $-$ curl curl $\bar{h}$ = grad ψ, where ψ is harmonic, and $\dfrac{\partial \psi}{\partial n} = 0$ on S, the transverse term being tangential to the boundary surface. In a simply-connected region, the only solution satisfying these requirements is the trivial one $\psi = 0$. In doubly-connected Region III, ψ must be a multiple of ψ_0, and $-$ curl curl $\bar{h}$ a multiple of $\bar{h}_0$. But $-$ curl curl $\bar{h}$ must simultaneously be functionally orthogonal to $\bar{h}_0$. The zero vector is the only one to have that property, so that $-$ curl curl $\bar{f}$ vanishes.

5. The magnetic splitting can be put in the form

$$\bar{a} = \bar{h}_0 \iiint \bar{a} \cdot \bar{h}_0 \, dV + \text{grad } \psi + \text{curl } \bar{v} \tag{28}$$

where $\bar{v}$ is perpendicular to the boundary and ψ is single-valued. Conversely, if a splitting such as Eq. 28 has been achieved, it is the magnetic one. Taking the divergence of both members of Eq. 28 shows, indeed, that ψ must satisfy

$$\nabla^2 \psi = \text{div } \bar{a} \qquad \text{while} \qquad \frac{\partial \psi}{\partial n} = a_n.$$

But these are exactly the relations satisfied by div $\bar{h}$. In consequence, ψ and div $\bar{h}$ differ by a constant (the only single-valued solution of the homogeneous problem), and the gradients of these two functions are identical.

C. Eigenvector Expansions Suitable for the Magnetic Splitting

The considerations valid for the electric splitting can be duplicated here. In simply connected regions, the eigenvectors relative to the integral operator $\iiint_V \mathcal{G}_H(\bar{r}, \bar{r}') \, dV'$ form a complete set. These eigenvectors fall into two categories:

(*a*) solenoidal eigenvectors $\bar{h}_m$, solutions of

$$-\text{curl curl } \bar{h}_m + \nu_m' \bar{h}_m = 0 \qquad \bar{u}_n \times \text{curl } \bar{h}_m = 0 \qquad \text{on } S. \tag{29}$$

The normalized $\bar{h}_m$ is simply related to the normalized $\bar{e}_m$ encountered in the electric problem. The relation is $\bar{h}_m = \dfrac{1}{\sqrt{\lambda_m'}} \text{curl } \bar{e}_m$, and both $\bar{e}_m$ and $\bar{h}_m$ have the same eigenvalues $\nu_m' = \lambda_m'$.

(*b*) irrotational eigenvectors $\bar{g}_m = \text{grad } \psi_m$, obtained from the eigenfunctions ψ_m of

$$\nabla^2 \psi_m + \nu_m'' \psi_m = 0 \qquad \frac{\partial \psi_m}{\partial n} = 0 \qquad \text{on } S. \tag{30}$$

These eigenvectors are all orthogonal to each other, and it will be assumed that they have been normalized. Their eigenvalues are different from zero. In Region III, the vector $\bar{h}_0$ must be added to obtain a set that is complete with respect to the space of square integrable vectors.

The expansion of the various terms of the magnetic splitting in terms of the $\bar{h}_0$, the $\bar{h}_m$ and the $\bar{g}_m$ is expressed by the important formula:

$$\bar{a} = \underbrace{\bar{h}_0 \iiint_V \bar{a}\cdot\bar{h}_0\, dV}_{\text{Term present in Region III only}} + \underbrace{\sum_m \vartheta_m \left[\iint_S (\bar{u}_n\cdot\bar{a})\psi_m dS - \iiint_V \psi_m \operatorname{div}\bar{a}\, dV \right]}_{\text{grad div } \bar{h}} + \underbrace{\sum_m \frac{\bar{h}_m}{\lambda_m'} \iiint \operatorname{curl}\bar{h}_m\cdot\operatorname{curl}\bar{a}\, dV.}_{-\operatorname{curl}\operatorname{curl}\bar{h}} \quad (31)$$

The convergence is in the mean, and, at a boundary point, the series converges to the tangential component of $\bar{a}$, all vectors of the expansion being tangential to the boundary surface.

The information obtained from the electric and the magnetic splitting is summarized in Table I. For each category of vectors, the

TABLE I.

Category	Region I	Region II	Region III
Any continuous vector $\bar{a}$	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta$	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t} + \langle\bar{a}, \bar{f}_0\rangle\bar{f}_0$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta$	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta + \langle\bar{a}, \bar{h}_0\rangle\bar{h}_0$
Solenoidal vector ($\operatorname{div}\bar{a} = 0$)	$\operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta$ (Ω harmonic)	$\operatorname{curl}\bar{t} + \langle\bar{a}, \bar{f}_0\rangle\bar{f}_0$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta$ (Ω harmonic)	$\operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \operatorname{curl}\vartheta + \langle\bar{a}, \bar{h}_0\rangle\bar{h}_0$ (Ω harmonic)
Solenoidal vector tangent to boundary	$\operatorname{curl}\bar{t}$ $\operatorname{curl}\vartheta$	$\operatorname{curl}\bar{t}$ $\operatorname{curl}\vartheta$	$\operatorname{curl}\bar{t}$ $\operatorname{curl}\vartheta + \langle\bar{a}, \bar{h}_0\rangle\bar{h}_0$
Irrotational vector	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega$	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t} + \langle\bar{a}, \bar{f}_0\rangle\bar{f}_0$ $\operatorname{grad}\Omega$	$\operatorname{grad}\varphi + \operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \langle\bar{a}, \bar{h}_0\rangle\bar{h}_0$
Irrotational vector perpendicular to boundary	$\operatorname{grad}\varphi$ $\operatorname{grad}\psi$	$\operatorname{grad}\varphi + \langle\bar{a}, \bar{f}_0\rangle\bar{f}_0$ $\operatorname{grad}\psi$	$\operatorname{grad}\varphi$ $\operatorname{grad}\psi$
Sourceless vector	$\operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega$ (Ω harmonic)	$\operatorname{curl}\bar{t} + \langle\bar{a}, \bar{f}_0\rangle\bar{f}_0$ $\operatorname{grad}\Omega$ (Ω harmonic)	$\operatorname{curl}\bar{t}$ $\operatorname{grad}\Omega + \langle\bar{a}, \bar{h}_0\rangle\bar{h}_0$ (Ω harmonic)

electric splitting is shown on the first line, the magnetic on the second, and it should be remembered that each of these splittings can be realized in one and only one way. The symbols Ω, φ, ψ denote, respectively, some scalar function, a function which vanishes at the boundary, and a function which takes some constant value on each surface of which the boundary is comprised. The symbols $\bar{v}$ and $\bar{t}$ denote respectively a

vector which is perpendicular to the boundary, and a vector which is tangential to the latter. The notation $\langle \bar{a}, \bar{f}_0 \rangle$ stands for the functional projection of $\bar{a}$ on $\bar{f}_0$, that is, $\iiint \bar{a} \cdot \bar{f}_0 \, dV$. The Ω, φ and ψ are all single-valued. For irrotational and sourceless vectors in Region III, the projection $\langle \bar{a}, \bar{h}_0 \rangle$ is proportional to the circulation of $\bar{a}$ around the hole.

IV. THE MIXED SPLITTING

It consists of splitting a vector as

$$\bar{a} = \text{grad } \varphi + \text{curl } \bar{v} + \bar{\alpha} \tag{32}$$

where φ vanishes on the boundary, $\bar{v}$ is perpendicular to the latter, and $\bar{\alpha}$ denotes a certain sourceless vector. To obtain this splitting, one can first calculate grad φ, which is a term furnished by the electric splitting, and then apply the magnetic splitting to what is left when grad φ is subtracted from $\bar{a}$. The mixed splitting is unique. The relative importance of the three terms can best be illustrated by writing down their expansions.

$$\bar{a} = \overbrace{\sum_m \bar{f}_m \left[- \iiint \varphi_m \, \text{div } \bar{a} \, dV \right]}^{\text{grad } \varphi}$$

(perpendicular to the boundary)

$$+ \overbrace{\sum_m \frac{\bar{h}_m}{\lambda_{m'}} \iiint \text{curl } \bar{h}_m \cdot \text{curl } \bar{a} \, dV}^{\text{curl } \bar{v}} + \bar{\alpha} \tag{33}$$

(tangential to the boundary)

which indicates which terms disappear for a solenoidal or an irrotational vector. The term $\bar{\alpha}$, which appears as a rest term in Eq. 32, can be expressed as

$$\bar{\alpha} = \sum_m \bar{g}_m \iint_S \left(a_n - \frac{\partial \varphi}{\partial n} \right) \psi_m dS + \bar{h}_0 \iiint_V \bar{a} \cdot \bar{h}_0 \, dV. \tag{34}$$

The term $- \bar{h}_0 \iiint \text{grad } \varphi \cdot \bar{h}_0 \, dV$ which the application of Eq. 31 introduces in Region III can be discarded. This term vanishes because $\iiint \text{grad } \varphi \cdot \text{grad } \psi_0 \, dV$ is equal to zero, as seen from an application of Eq. 5 where A would be the single-valued function φ, and B the harmonic function ψ_0. Notice that φ vanishes for a solenoidal vector,

and that the term $\frac{\partial \varphi}{\partial n}$ appearing in Eq. 34 can be put equal to zero in those circumstances.

Another expression is possible for $\bar{a}$:

$$\bar{a} = \bar{f}_0 \left[\iint (\bar{u}_n \cdot \bar{a}) \varphi_0 dS - \iiint \varphi_0 \operatorname{div} \bar{a}\, dV \right] - \sum_m \frac{\bar{e}_m}{\lambda_m'} \iint_S [\bar{u}_n \times (\bar{a} - \operatorname{curl} \bar{v})] \cdot \operatorname{curl} \bar{e}_m dS. \quad (35)$$

Notice that $\bar{v}$ vanishes for an irrotational vector. It is sometimes desirable to calculate φ directly, instead of finding its gradient by a formula such as Eq. 33. The steps are rather trivial, and involve expansion in the normalized[5] eigenfunctions φ_m' of Eq. 19. The final result is:

$$\varphi = - \sum_m \frac{\varphi_m'}{\lambda_m''} \iiint \varphi_m' \operatorname{div} \bar{a}\, dV = - \sum_m \varphi_m \iiint \varphi_m \operatorname{div} \bar{a}\, dV \quad (36)$$

where φ_m is the scalar potential used above for the normalized vectors $\bar{f}_m = \operatorname{grad} \varphi_m$. Similarly, it is interesting to have an expansion for $\bar{v}$ in terms of the eigenvectors of the electric splitting. Notice that $\bar{v}$ stands for $-\operatorname{curl} \bar{f}$, and is consequently solenoidal. This implies that the $\bar{f}_m$ do not appear in the expansion. Notice also that $\bar{v}$, being the curl of minus $\bar{f}$, does not have any functional projection on $\bar{f}_0$, because $\bar{f}_0$ cannot be represented as a curl. In consequence, the terms in $\bar{e}_m$ are the only ones to survive in the expansion of $\bar{v}$, which then turns out to be

$$\bar{v} = \sum_m \frac{\bar{e}_m}{\lambda_m'} \iiint \bar{e}_m \cdot \operatorname{curl} \bar{a}\, dV. \quad (37)$$

Obviously, any gradient perpendicular to the boundary can be added to Eq. 37, and still furnish an acceptable $\bar{v}$, that is, yield an unmodified transverse term curl $\bar{v}$. The $\bar{v}$ given by Eq. 37 is the particular choice associated with the magnetic splitting.

REFERENCES

(1) T. EICHMANN AND E. P. WIGNER, "Electromagnetic Field Expansion in Loss-Free Cavities Excited Through Holes," *J. Appl. Phys.*, Vol. 24, pp. 262–267 (1953).

(2) A. F. STEVENSON, "Note on the Existence and Determination of a Vector Potential," *Quart. Appl. Math.*, Vol. XII, pp. 194–198 (1954).

[5] Notice that the ϕ_m appearing in formulas 22 and 33 are not normalized. It is their gradient $\bar{f}_m$ which is normalized, and an application of Eq. 5 shows that $\iiint \varphi_m^2 dV = \frac{1}{\lambda_m''}$. The functions ϕ_m' are, consequently, identical with $\sqrt{\lambda_m''}\varphi_m$.

(3) K. KUROKAWA, "The Expansions of Electromagnetic Fields in Cavities," *IRE Trans.*, Vol. MTT-6, pp. 178–187 (1958).
(4) H. WEYL, "Das Asymptotische Verteilungsgesetz der Eigenschwingungen Eines Beliebig Gestalteten Elastischen Korpers," *Rendic. Circ. Mat. Palmero*, Vol. 39, pp. 1–50 (1915).
(5) H. WEYL, "Uber die Randwertaufgabe der Strahlungstheorie und Asympotische Spektralgesetze," *J. Reine. Angew. Math.*, Vol. 143, pp. 177–202 (1913).
(6) F. RIESZ AND B. S. NAGY, "Lecons d'Analyse Fonctionnelle," Third edition, Paris, Gauthier-Villars, 1955, p. 239.
(7) C. E. WEATHERBURN, "Vector Integral Equations of the First Kind," *Quart. J. Pure Appl. Math.*, Vol. 46, pp. 334–356 (1915).
(8) C. E. WEATHERBURN, "Vector Integral Equations and Gibbs' Dyadics," *Trans. Cambr. Phil. Soc.*, Vol. XXII, No. VIII, pp. 133–158 (1916).
(9) H. WEYL, "Uber das Spektrum der Hohlraumstrahlung," *J. Reine. Angew. Math.*, Vol. 141, pp. 163–181 (1912).

APPENDIX

In this appendix, the properties of the eigenvectors of the Laplacian operator are examined for two different sets of boundary conditions. Only those eigenvectors relative to eigenvalues different from zero are considered.

(*a*) Consider first the eigenvectors $\bar{F}_m$ of

$$\text{grad div } \bar{F}_m - \text{curl curl } \bar{F}_m + \lambda_m \bar{F}_m = 0 \qquad \text{with} \left\{ \begin{array}{c} \bar{u}_n \times \bar{F}_m = 0 \\ \text{div } \bar{F}_m = 0 \end{array} \right\} \text{ on } S. \tag{38}$$

The eigenvector $\bar{F}_m$ can be split electrically as grad α_m + curl $\bar{w}_m$. No term in $\bar{f}_0$ is necessary in Region II, because $\bar{F}_m$ and $\bar{f}_0$ are functionally orthogonal to each other. This results from the formula

$$\iiint_V \bar{f}_0 \cdot \nabla^2 \bar{F}_m dV = -\lambda_m \iiint_V \bar{f}_0 \cdot \bar{F}_m dV$$

$$= \iiint_V \bar{F}_m \cdot \nabla^2 \bar{f}_0 dV + \iint_S [(\bar{u}_n \cdot \bar{f}_0) \text{ div } \bar{F}_m - (\bar{u}_n \cdot \bar{F}_m) \text{ div } \bar{f}_0$$

$$+ (\bar{u}_n \times \bar{f}_0) \cdot \text{curl } \bar{F}_m - (\bar{u}_n \times \bar{F}_m) \cdot \text{curl } \bar{f}_0] dS.$$

All integrals of the last member vanish, which implies that $\iiint_V \bar{f}_0 \cdot \bar{F}_m dV = 0$. Introducing grad α_m + curl $\bar{w}_m$ into Eq. 38, one arrives at the formula

$$\text{grad } [\nabla^2 \alpha_m + \lambda_m \alpha_m] = \text{curl } [\text{curl curl } \bar{w}_m - \lambda_m \bar{w}_m]. \tag{39}$$

Calling $\bar{g}$ the common value of the two members, one sees right away that $\bar{g}$ must be sourceless and perpendicular to the boundary surface. On this surface, indeed, α_m is zero, and div $\bar{F}_m = \nabla^2 \alpha_m$ vanishes too. The gradient of $\nabla^2 \alpha_m + \lambda_m \alpha_m$ is consequently perpendicular to that surface. In simply-bounded regions, there is no non-zero sourceless vector perpendicular to the boundary, and both members of Eq. 39 must vanish. This implies that $\nabla^2 \alpha_m + \lambda_m \alpha_m$ is constant throughout the volume, and that the constant is zero since this is the value $\nabla^2 \alpha_m + \lambda_m \alpha_m$ takes on the boundary. We write

$$\nabla^2 \alpha_m + \lambda_m \alpha_m = 0 \qquad \alpha_m = 0 \qquad \text{on } S. \tag{40}$$

But this implies also that the transverse part curl $\bar{w}_m$, which will now be written as $\bar{e}_m$, satisfies

$$-\text{curl curl } \bar{e}_m + \lambda_m \bar{e}_m = 0 \qquad \bar{u}_n \times \bar{e}_m = 0 \qquad \text{on } S. \tag{41}$$

One then arrives at the conclusion that to each eigenvalue λ_m there correspond two possible sorts of eigenvectors *viz:* grad α_m, where α_m satisfies Eq. 40, and (or) $\bar{e}_m$, where $\bar{e}_m$ satisfies Eq. 41. Conversely any eigenvector of Eq. 41, and the gradient of any eigenfunction of Eq. 40,

are eigenvectors of the original problem, Eq. 38. There is, consequently, complete identity between the latter and the ensemble of the grad α_m supplemented by the ensemble of the $\bar{e}_m$. The same is true for the eigenvalues λ_m, which are made up of the eigenvalues λ_m'' of Eq. 40 supplemented by the eigenvalues λ_m' of Eq. 41, some of the eigenvalues being possibly common to both classes.

In doubly bounded regions, the same sequence of arguments can be used. Vector $\bar{g}$ must be a multiple of $\bar{f}_0$, being sourceless and perpendicular to the boundary. But Eq. 39 indicates that $\bar{g}$ can be put in the form of the curl of a vector potential. This is impossible for $\bar{f}_0$ and its non-zero multiples, so $\bar{g}$ must vanish.

(*b*) Consider now the eigenvectors $\bar{H}_m$ of

$$\text{grad div } \bar{H}_m - \text{curl curl } \bar{H}_m + \nu_m \bar{H}_m = 0 \qquad \text{with} \left\{ \begin{matrix} \bar{u}_n \times \text{curl } \bar{H}_m = 0 \\ \bar{u}_n \cdot \bar{H}_m = 0 \end{matrix} \right\} \text{on } S. \tag{42}$$

The same sort of argument as in part (*a*) can be applied here. The eigenvector $\bar{H}_m$ can be split magnetically as grad β_m + curl $\bar{v}_m$, with no term in $\bar{h}_0$ needed in doubly connected regions. Introducing in Eq. 42, one obtains

$$\text{grad } [\nabla^2 \beta_m + \nu_m \beta_m] = \text{curl } [\text{curl curl } \bar{v}_m - \nu_m \bar{v}_m]. \tag{43}$$

The common value of both members is a sourceless vector. This vector is tangent to the boundary surface, because grad β_m is tangent, being the difference of the two tangential vectors $\bar{H}_m$ and curl $\bar{v}_m$, and because grad div $\bar{H}_m$ = grad $\nabla^2 \beta_m$ is tangent, being the difference of the two tangential vectors $-\nu_m \bar{H}_m$ and curl curl $\bar{H}_m$. In consequence, both members vanish in a simply connected region, where a sourceless tangential vector cannot exist. This implies that $\nabla^2 \beta_m + \nu_m \beta_m$ is equal to a constant c_m throughout the volume, or that $\nabla^2 \left(\beta_m - \frac{c_m}{\nu_m} \right) + \nu_m \left(\beta_m - \frac{c_m}{\nu_m} \right)$ vanishes throughout the volume. In other words, the longitudinal part can be put in the form grad ψ_m where

$$\nabla^2 \psi_m + \nu_m \psi_m = 0 \qquad \frac{\partial \psi_m}{\partial n} = 0 \qquad \text{on } S. \tag{44}$$

The transverse part $\bar{h}_m$ = curl $\bar{v}_m$ satisfies

$$-\text{curl curl } \bar{h}_m + \nu_m \bar{h}_m = 0 \qquad \bar{u}_n \times \text{curl } \bar{h}_m = 0 \qquad \text{on } S. \tag{45}$$

Conversely, each grad ψ_m from Eq. 44, and each $\bar{h}_m$ from Eq. 45, is an eigenvector of the original problem (42). Notice, by taking the curl of Eq. 45, that curl $\bar{h}_m$ satisfies the eigenvector equation defining $\bar{e}_m$, that is, Eq. 41. Conversely, curl $\bar{e}_m$ satisfies the eigenvector equation (45) defining the $\bar{h}_m$. This is seen by taking the curl of Eq. 41, and noticing that $\bar{u}_n \times \bar{e}_m = \bar{u}_n \times \text{curl} \left(\frac{\text{curl } \bar{e}_m}{\lambda_m} \right) = 0$ on S. *The solenoidal eigenvectors of both problems are multiples of the curl of each other.*

In a doubly connected region, both members of Eq. 43 are multiples of $\bar{h}_0$, or vanish. But β_m, and $\nabla^2 \beta_m + \nu_m \beta_m$ are single-valued functions, a property of the magnetic splitting. The integral of their gradient around the hole vanishes. The integral of $\bar{h}_0$, however, is different from zero, and the only way to resolve this inconsistency is for grad $[\nabla^2 \beta_m + \nu_m \beta_m]$ to vanish. From there on, the steps follow as for simply connected regions.

STRESSES IN CERTAIN THIN PLATES ROTATING ABOUT AN AXIS LYING IN THEIR MIDDLE PLANE

BY

R. S. DHALIWAL [1]

ABSTRACT

In this paper N. I. Muskhelishvili's method has been employed for solving problems of certain thin elastic plates having curvilinear boundaries and rotating steadily about an axis lying in the middle plane of the plate. Considered in detail is the problem of a plate having a regular curvilinear polygonal boundary, in general. The results for a circular, dumbbell-shaped, cycloidal and cogwheel-shaped plates have been derived as particular cases. The numerical results are presented in the form of tables and graphs.

NOMENCLATURE

The following nomenclature is used in this paper:

x, y = rectangular coordinates
$z = x + iy$
ω = constant angular velocity
$\zeta = \rho e^{i\theta}$
$\sigma = e^{i\theta}$, namely the value of ζ at the boundary
V = body force potential
xx, yy, xy = stress components
$\rho\rho, \theta\theta, \rho\theta$ = stress components in curvilinear coordinates
u, v = displacements
$\varphi(z), \psi(z)$ = analytic functions of z
$\varphi(\zeta), \psi(\zeta)$ = analytic functions of ζ
$g(\zeta)$ = mapping function
$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$
ϵ = density
λ, μ = Lamé constants
E = Young's modulus
η = Poisson's ratio
$\gamma = 2(1 - \eta)$
γ_0 = boundary of unit circle.

INTRODUCTION

The problems of steady rotation of plates about normal axes have been solved by Love (1),[2] Timoshenko and Goodier (2), Mindlin (3) and

[1] Department of Applied Sciences, Curu Nanak Engineering College, Ludhiana. Presently with Department of Mathematics, Indian Institute of Technology, Kharagpur, India.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

Sen (4). The stresses set up by the rotation of the plate in the form of an ellipse are analyzed by Stevenson (5) by using the complex variable method. Solutions for plates in the form of a cardioid and epitrochoid have been obtained by Mitra (6) and Madan Mohan (7), respectively, by using Muskhelishvili's method.

The problems of a circular and a triangular plate rotating about a diameter and a side respectively have been solved by Hodge (8) by using a proper stress function. Sen (9) has given a direct method for solving plane elasticity problems, which has been further developed by Sen Gupta (10). Recently Arkilic (11) has applied the same method to solve the problem of curvilinear polygonal plates rotating about an axis lying in the middle plane of plates.

To the author's knowledge, Muskhelishvili's method has not been employed by any investigator to find stresses set up in plates rotating steadily about an axis lying in the middle plane of the plate. The problem has been considered as a statical one and inertia forces have been treated as body forces. The stress and displacement components at a point are expressed in terms of two analytic functions and a body force potential. The elastic region is mapped into the interior of a unit circle and under given boundary conditions a functional equation is obtained. Using function theory the complex functions are determined from this functional equation and its complex conjugate.

FUNDAMENTAL EQUATIONS

The stress and displacement components can be put in terms of two analytic functions $\phi(z)$, $\psi(z)$ and the body force potential V as

$$\left.\begin{aligned} xx + yy &= 2[\varphi'(z) + \bar{\varphi}'(\bar{z})] + \frac{2(\lambda+\mu)\epsilon}{\lambda+2\mu} V, \\ yy - xx + 2ixy &= 2[\bar{z}\varphi''(z) + \psi'(z)] - \frac{8\mu\epsilon}{\lambda+2\mu}\frac{\partial^2 V_1}{\partial z \partial \bar{z}}, \end{aligned}\right\} \tag{1}$$

$$2\mu(u + i\vartheta) = \frac{\lambda+3\mu}{\lambda+\mu}\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) + \frac{4\mu\epsilon}{\lambda+2\mu}\frac{\partial V_1}{\partial z}, \tag{2}$$

where V_1 is the particular integral of $\nabla^2 V_1 = V$. The boundary condition for the unstressed boundary may be put in the form

$$\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z}) = \gamma\epsilon\frac{\partial V_1}{\partial \bar{z}} - \epsilon\int V dz. \tag{3}$$

Equations 1, 2 and 3 are given by Stevenson (5), with a slightly changed notation.

We introduce the mapping function, which maps the region occupied by the plate onto the unit circle in the ζ-plane,

$$z = g(\zeta). \tag{4}$$

Curvilinear coordinates (ρ, θ) are thus introduced into the z-plane which are the maps of the polar coordinates in the ζ-plane as given by $\zeta = \rho e^{i\theta}$. The stress and displacement components with reference to the new coordinates (ρ, θ) may be derived from Eqs. 1 and 2; the results are as follows:

$$\left.\begin{aligned}
\rho\rho + \theta\theta &= 2[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta})] + \frac{2(\lambda+\mu)\epsilon}{\lambda+2\mu} V, \\
\theta\theta - \rho\rho + 2i\rho\theta &= \frac{2\zeta^2}{\rho^2 \bar{g}'(\bar{\zeta})}[\bar{g}(\bar{\zeta})\Phi'(\zeta) + g'(\zeta)\Psi(\zeta)] \\
&\quad - \frac{8\mu\epsilon}{\lambda+2\mu} \frac{\partial^2 V_1}{\partial\zeta\partial\bar{\zeta}} \frac{1}{g'(\zeta)\bar{g}'(\bar{\zeta})}, \\
2\mu(u_\rho + iu_\theta) &= \frac{\bar{\zeta}}{\rho} \frac{\bar{g}'(\bar{\zeta})}{|g'(\zeta)|}\left[\frac{\lambda+3\mu}{\lambda+\mu}\varphi(\zeta) - \frac{g(\zeta)}{\bar{g}'(\bar{\zeta})}\bar{\varphi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta})\right] \\
&\quad + \frac{4\mu\epsilon}{\lambda+2\mu} \frac{\partial V_1}{\partial\zeta} \frac{1}{\bar{g}'(\bar{\zeta})},
\end{aligned}\right\} \tag{5}$$

where

$$\begin{aligned}
\varphi(z) &= \varphi[g(\zeta)] \equiv \varphi(\zeta), \quad & \psi(z) &= \psi[g(\zeta)] \equiv \psi(\zeta), \\
\Phi(\zeta) &= \varphi'(\zeta)/g'(\zeta), & \Psi(\zeta) &= \psi'(\zeta)/g'(\zeta).
\end{aligned}$$

If the plate is rotating about the y-axis with angular velocity ω, the body force potential is represented by

$$V = -\tfrac{1}{2}\omega^2 x^2 = -\tfrac{1}{8}\omega^2(z+\bar{z})^2 = -\tfrac{1}{8}\omega^2[g(\zeta) + \bar{g}(\bar{\zeta})]^2. \tag{6}$$

The boundary equation, Eq. 3, is transformed into

$$\varphi(\sigma) + \frac{g(\sigma)}{\bar{g}'(\bar{\sigma})}\bar{\varphi}'(\bar{\sigma}) + \bar{\psi}(\bar{\sigma}) = F(\sigma), \tag{7}$$

where $\sigma = e^{i\theta}$ denotes the value of ζ on the boundary γ_0 and

$$\begin{aligned}
F(\sigma) = &-\frac{\gamma\epsilon}{96}\omega^2[g^3(\sigma) + 3g^2(\sigma)\bar{g}(\bar{\sigma}) + 3g(\sigma)\bar{g}^2(\bar{\sigma})] \\
&+ \frac{\epsilon}{8}\omega^2 \int [g(\sigma) + \bar{g}(\bar{\sigma})]^2 g'(\sigma) d\sigma.
\end{aligned} \tag{8}$$

STRESSES IN REGULAR CURVILINEAR POLYGONAL PLATES OF $(n-1)$ SIDES ROTATING ABOUT y-AXIS $(n > 3)$

The region bounded by

$$x = R(\cos\theta + m\cos n\theta), \qquad y = R(\sin\theta + m\sin n\theta), \tag{9}$$

where $R > 0$, $0 \leq m \leq \frac{1}{n}$ and n is an integer greater than unity, is mapped conformally on a unit circle bounded by γ_0 by the transformation

$$z = g(\zeta) = R(\zeta + m\zeta^n). \tag{10}$$

Curves (9) are, in general, some regular curvilinear polygons of $(n-1)$ sides having n rounded corners provided

$$\frac{n-1}{n} \leq 1 - m \leq r \leq 1 + m \leq \frac{n+1}{n},$$

where

$$r^2 = x^2 + y^2.$$

Now $\bar{\sigma} = \frac{1}{\sigma}$ and from Eqs. 8 and 10 we find that for $n > 2$

$$F(\sigma) = A_1\sigma^{2n} + A_2\sigma^{2n+1} + A_3\sigma^{2n-1} + A_4\sigma^{n+2} + A_5\sigma^n + A_6\sigma^{n-2} + A_7\sigma^3 + A_8\sigma + A_9\sigma^{-1} + A_{10}\sigma^{2-n} + A_{11}\sigma^{-n} + A_{12}\sigma^{1-2n} \tag{11}$$

where

$$\left.\begin{aligned}
A_1 &= \frac{1}{3}\left(1 - \frac{\gamma}{4}\right) Pm^3,\\
A_2 &= \left(1 - \frac{4}{\gamma}\right) Pm^2,\\
A_3 &= \left(\frac{2n}{2n-1} - \frac{\gamma}{4}\right) Pm^2,\\
A_4 &= \left(1 - \frac{\gamma}{4}\right) Pm,\\
A_5 &= \left[\frac{2}{n}(1 + n + nm^2) - (m^2 + 2)\frac{\gamma}{4}\right] Pm,\\
A_6 &= \left(\frac{n}{n-2} - \frac{\gamma}{4}\right) Pm,\\
A_7 &= \frac{1}{3}\left(1 - \frac{\gamma}{4}\right) P,
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
A_8 &= \left[2(1 + m^2 + nm^2) - (1 + 2m^2)\frac{\gamma}{4}\right]P,\\
A_9 &= -\left[(1 + 2nm^2) + (1 + 2m^2)\frac{\gamma}{4}\right]P,\\
A_{10} &= -\left(\frac{2}{n-2} + \frac{\gamma}{4}\right)Pm,\\
A_{11} &= -\left[\frac{1}{n}(2 + nm^2) + (m^2 + 2)\frac{\gamma}{4}\right]Pm,\\
A_{12} &= -\left(\frac{1}{2n-1} + \frac{\gamma}{4}\right)Pm,\\
P &= \frac{\epsilon}{8}\omega^2 R^3.
\end{aligned}\right\} \quad (12\ \textit{Cont.})$$

The complex functions $\varphi(\zeta)$, $\psi(\zeta)$ are holomorphic within the unit circle, and we can assume power series expansion for these functions as

$$\varphi(\zeta) = \sum_1^\infty a_p\zeta^p, \qquad \psi(\zeta) = \sum_1^\infty b_q\zeta^q, \tag{13}$$

where the coefficients a_p, b_q are complex constants.

Multiplying both sides of Eq. 7 by $d\sigma/2\pi i(\sigma - \zeta)$ and integrating over the unit circle γ_0 we obtain

$$\begin{aligned}
&\varphi(\zeta) + \bar{b}_0 + (1 - m^2n)2\bar{a}_2 + (1 - m^2n)\bar{a}_1\zeta\\
&\qquad + \sum_{p=0}^{n}(n + 1 - p)\bar{a}_{n+1-p}\zeta^p = A_1\zeta^{3n} + A_2\zeta^{2n+1}\\
&\qquad + A_3\zeta^{2n-1} + A_4\zeta^{n+2} + A_5\zeta^n + A_6\zeta^{n-2} + A_7\zeta^3 + A_8\zeta.
\end{aligned} \tag{14}$$

The quantities A_k being real, constants a_p and b_q will also be real. Comparison of like powers of ζ on the two sides of Eq. 14 leads to a system of linear equations from which we find that only the following real coefficients a_k are different from zero

$$\left.\begin{aligned}
a_1 &= (A_8 - mnA_5)/2(1 - m^2n),\\
a_3 &= [m(n-2)A_6 - A_7]/[3m^2(n-2) - 1],\\
a_{n-2} &= (3mA_7 - A_6)/[3m^2(n-2) - 1],\\
a_n &= [(2 - m^2n)A_5 - mA_8]/2(1 - m^2n),\\
a_{n+2} &= A_4, \quad a_{2n-1} = A_3, \quad a_{2n+1} = A_2, \quad a_{3n} = A_1.
\end{aligned}\right\} \tag{15}$$

Hence for $n > 2$

$$\varphi(\zeta) = a_1\zeta + a_3\zeta^3 + a_{n-2}\zeta^{n-2} + a_n\zeta^n + a_{n+2}\zeta^{n+2} + a_{2n-1}\zeta^{2n-1} + a_{2n+1}\zeta^{2n+1} + a_{3n}\zeta^{3n}. \quad (16)$$

Similarly from the conjugate complex of Eq. 7 we obtain

$$\psi(\zeta) = A_{12}\zeta^{2n-1} + A_{11}\zeta^n + A_{10}\zeta^{n-2} + A_9\zeta + [mna_n + (1 - m^2n)a_1]\zeta^{-1} + m(n-2)a_{n-2}\zeta^{-3} + 3ma_3\zeta^{2-n} + ma_1\zeta^{-n} - (m + \zeta^{n-1})\varphi'(\zeta)/\zeta^n(1 + mn\zeta^{n-1}). \quad (17)$$

The functions $\varphi(\zeta)$, $\psi(\zeta)$ being determined, the stress distribution can be obtained from Eqs. 5. We have

$$[\theta\theta]_{\zeta=\sigma} = \frac{\epsilon\omega^2R^2}{[1+m^2n^2+2mn\cos(n-1)\theta]} \times\Bigg[\frac{1}{8(1-m^2n)}\{1-\eta-m^2[(n^2-8n+4)-\eta(n^2-4n)] - m^4n[(4n^2-2n+1)-\eta(2n+1)]-m^6n^3(1-\eta)\} + \frac{m}{[3m^2(n-2)-1]}[m(3-n)\cos 2\theta+(3m^2n-1)\cos(n-3)\theta] + \frac{\gamma}{4}m(1+m^2n)\cos(n-1)\theta+\frac{\gamma}{8}m^2\cos(2n-2)\theta\Bigg]. \quad (18)$$

The following particular cases of interest have been discussed in detail:

(*a*) *Stresses in a Circular Plate Rotating about a Diameter*

For this case $m = 0$, so $z = R_1\zeta$ and we get

$$[\theta\theta]_{\zeta=\sigma} = \tfrac{1}{8}(1 - \eta)\epsilon\omega^2R_1^2, \quad (19)$$

where $R_1 = R$ is the radius of the circle.

The analytic functions, for this case, are given by

$$\varphi(\zeta) = A\zeta + A_7\zeta^3, \qquad \psi(\zeta) = (B - 3A)\zeta,$$

where

$$A = \left(1 - \frac{\gamma}{8}\right)P, \qquad B = -\left(1 + \frac{\gamma}{4}\right)P.$$

The stress and displacement components at any point of the plate may be obtained from Eqs. 5. We have

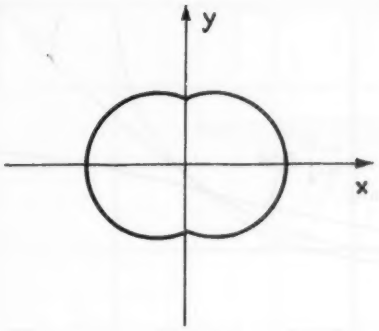


FIG. 1.

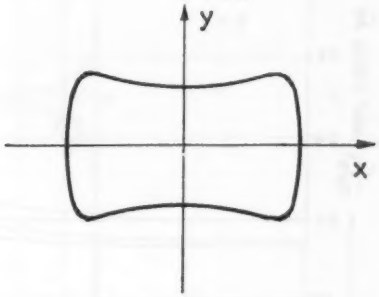


FIG. 2.

$$\rho\rho = \tfrac{1}{4}\epsilon\omega^2\left(\frac{3+\eta}{4} + \cos 2\theta\right)(R_1^2 - r^2),$$

$$\theta\theta = \tfrac{1}{4}\epsilon\omega^2\left[\frac{3+\eta}{4}R_1^2 - \frac{1+3\eta}{4}r^2 - (R_1^2 - r^2)\cos 2\theta\right],$$

$$\rho\theta = \tfrac{1}{4}\epsilon\omega^2(r^2 - R_1^2)\sin 2\theta,$$

$$u_\rho = \frac{\epsilon\omega^2}{4E}\left[\{\tfrac{1}{4}(3+\eta)(1-\eta) + (1+\eta)\cos 2\theta\}rR_1^2 - \left(\frac{1-\eta^2}{4} + \frac{1+\eta}{3}\cos 2\theta\right)r^3\right],$$

$$u_\theta = \frac{\epsilon\omega^2}{12E}(1+\eta)(3rR_1^2 - 2r^3)\sin 2\theta.$$

The solution obtained agrees with that of Hodge (**8**) who solved the problem by a different method.

(*b*) *Boundary Stresses for Two Cusped Epicycloidal Plates*

For this case $n = 3$, $m = \frac{1}{3}$, Fig. 1, and we get

$$[\theta\theta]_{r=a} = \frac{1}{36}(1-\eta)\epsilon\omega^2R^2(3 + \cos 2\theta). \tag{20}$$

At the cusps, where $\theta = \pm\dfrac{\pi}{2}$ this hoop stress is minimum and equal to half the maximum hoop stress which occurs at $\theta = 0$ or π.

TABLE I.

m \ θ	0°	15°	30°	45°	60°	75°	90°
1/4	0.1274	0.1299	0.1406	0.1732	0.2785	0.7608	3.079
1/6	0.1426	0.1467	0.1622	0.2011	0.2982	0.5371	0.802
1/12	0.1938	0.1995	0.2185	0.2582	0.3181	0.3959	0.4374

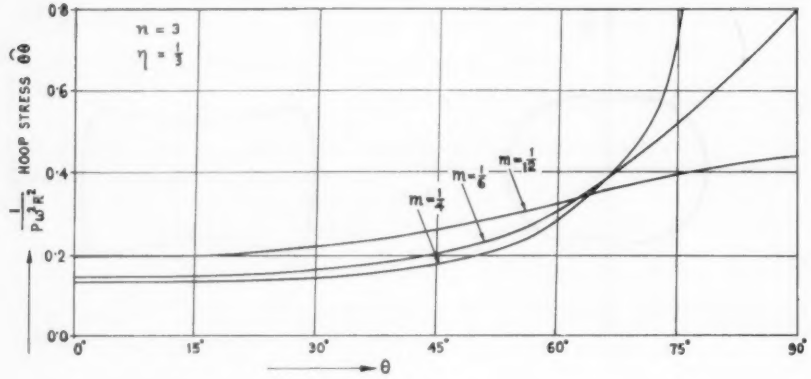


FIG. 3.

(*c*) *Boundary Stresses for Dumbbell-Shaped Plates*

For this case $n = 3$, $m < \frac{1}{3}$, Fig. 2. Taking $\eta = \frac{1}{3}$, the values of $[\theta\theta/\epsilon\omega^2R^2]_{r=a}$ in this case for different values of m and θ are given in Table I and are shown graphically in Fig. 3.

(*d*) *Boundary Stresses for Cogwheel-Shaped Plates*

For $n = 21$, $m < \dfrac{1}{21}$, the plates takes the approximate shape of a cog-wheel with twenty teeth (shallow teeth), Fig. 4.

Taking $\eta = \frac{1}{3}$, the values of $[\theta\theta/\epsilon\omega^2R^2]_{r=a}$ in this case for different values of m and θ are given in Table II and are shown graphically in

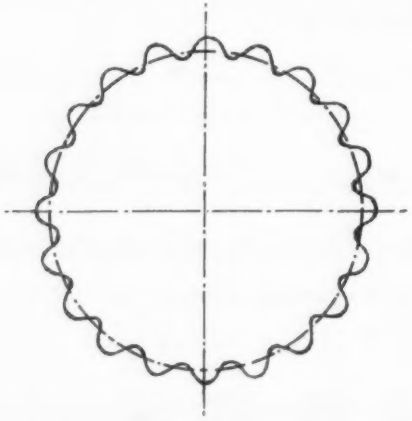


FIG. 4.

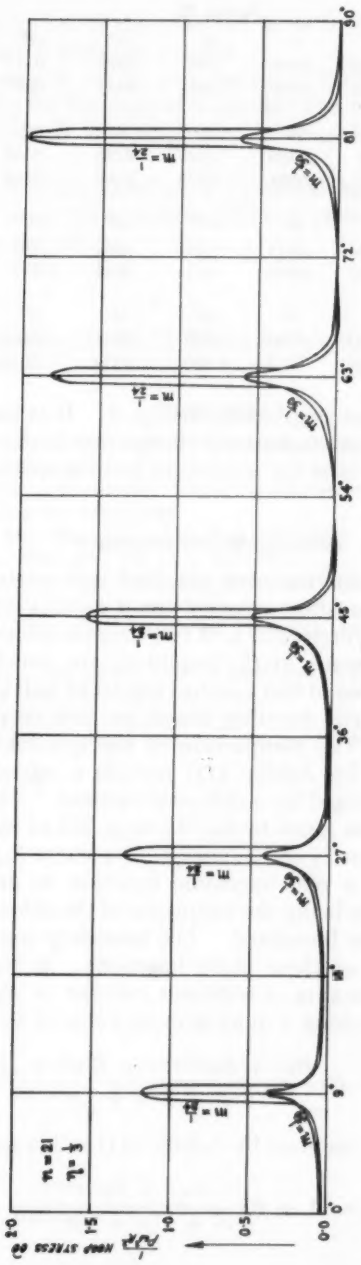


FIG. 5.

TABLE II.

m \ θ	0°	3°	6°	9°	12°	15°	18°	21°
1/30	.0426	.0472	.0785	.386	.0558	.0371	.0390	.0502
1/24	.0367	.0395	.0590	1.180	.0334	.0287	.0326	.0419
m \ θ	24°	27°	30°	33°	36°	39°	42°	45°
1/30	.0961	.434	.0366	.0237	.0296	.0450	.101	.511
1/24	.0756	1.330	.0084	.0135	.0218	.0349	.0768	1.580
m \ θ	48°	51°	54°	57°	60°	63°	66°	69°
1/30	.0283	.0122	.0179	.0335	.0934	.587	.0340	.0070
1/24	−.0062	−.0002	.0085	.0211	.0621	1.827	−.0050	−.0072
m \ θ	72°	75°	78°	81°	84°	87°	90°	
1/30	.0085	.0202	.0742	.635	.0515	.0100	.0049	
1/24	−.0022	.0060	.0372	1.980	.0114	−.0048	−.0063	

Table II and are shown graphically in Fig. 5. It is found that general character of the hoop stress does not change due to change in the values of m. For all values of m the maximum hoop stresses are occurring at the same places.

CONCLUSIONS AND DISCUSSION

In this paper solution has been obtained for certain thin curvilinear plates rotating with constant velocity about an axis lying in the middle plane of the plate. Forcircular and two-cusped epicycloidal plates the magnitudes of the stresses at the boundary are found, by comparison of the results with those of Sen Gupta (10), to be half of the magnitudes for the same boundaries rotating about an axis perpendicular to the plane of the plate. The comparison of the general results of Sec. 2 with those obtained by Arkilic (11) reveals a mistake in his results which have been obtained by a different method. The general expression for the hoop-stress given by Eq. 12 on p. 283 of his paper lacks two terms of the type $S_1 \cos 2\eta$ and $S_2 \cos (n-2)\eta$ where S_1, S_2 are constants.

He has assumed a real harmonic function Ω_0 and two harmonic functions F and G, one being the conjugate of the other, to satisfy conditions on the stress-free boundary. The boundary conditions have been formulated in terms of these three functions. It may be mentioned that, to find the constants, a sufficient number of algebraic equations will be obtained by taking a more general value of Ω_0 as,

$$\Omega_0 = A + Re\,\frac{Be^{\zeta} + De^{(2n+1)\zeta} + B'e^{3\zeta} + D'e^{(n-1)\zeta}}{e^{\zeta} + \lambda(n+1)e^{(n+1)\zeta}} \qquad (a)$$

than the value of Ω_0 assumed by Arkilic (11) in his paper as

$$\Omega_0 = A + Re\,\frac{Be^{\zeta} + De^{(2n+1)\zeta}}{e^{\zeta} + \lambda(n+1)e^{(n+1)\zeta}}. \qquad (b)$$

On the assumption of the value of Ω_0 as in (*a*) the results with his method would agree with our results.

The coefficients of $\cos 2\theta$ and $\cos (n-3)\theta$ in our expression (18) vanish identically for the case of a circular and two-cusped epicycloidal plates which explains the similarity of boundary stresses for these particular cases.

In the case of a cogwheel or a dumbbell his results for boundary stresses are not even approximate which is clear from the comparison of corresponding graphs.

Acknowledgment

The author thanks Dr. D. N. Mitra for his kind help in the preparation of this paper.

REFERENCES

(1) A. E. H. LOVE, "Mathematical Theory of Elasticity," fourth edition, Cambridge University Press, 1927, p. 146.
(2) S. TIMOSHENKO AND J. N. GOODIER, "Theory of Elasticity," second Edition, New York, McGraw-Hill Book Co., 1951, p. 69.
(3) R. D. MINDLIN, *Phil. Mag.*, Ser. 7, Vol. 26, p. 713 (1938).
(4) B. SEN, *Bull. Cal. Math. Soc.*, Vol. 36, p. 58 (1944).
(5) A. C. STEVENSON, *Phil. Mag.*, Ser. 7, Vol. 34, p. 766 (1943).
(6) D. N. MITRA, *ZAMP*, Vol. 6, p. 136 (1955).
(7) MADAN MOHAN, "Proc. Sec. Congress on Theoretical and App. Mech. New Delhi," 1956, p. 95.
(8) P. G. HODGE, JR., *J. App. Mech., Trans. ASME*, Vol. 23, p. 273 (1956).
(9) B. SEN, *Phil. Mag.*, Ser. 7, Vol. 39, p. 992 (1948).
(10) A. M. SEN GUPTA, *ZAMM*, Vol. 37, p. 27 (1957).
(11) G. M. ARKILIC, JOUR. FRANKLIN INST., Vol. 266, pp. 279–292 (1958).

SOME CONSIDERATIONS OF THE MECHANICS AND DESIGN LIMITATIONS OF AUTOFRETTAGE

BY

J. H. FAUPEL [1]

ABSTRACT

Results reported on heavy-wall steel cylinders prestressed by autofrettage (plastic radial expansion) are discussed. Optimum design is considered relative to flow, section size, and properties of the material. It is shown that the ability of steel to accept and retain residual stress is a function of the diameter ratio and strength of the material. Design procedures are suggested.

NOMENCLATURE

A_o—original cross-sectional area, sq. in.
A—cross-sectional area, sq. in.
a_0—bore radius, in. (initial value)
a—bore radius, in.
b—interface radius, in.
b'—reverse yield boundary radius, in.
c_o—original O.D. radius, in.
c—O.D. radius, in.
d—differential operator
E—modulus of elasticity, psi.
K—strength coefficient, psi.
k—$\sigma_i/\sqrt{3}$, psi.
ln—symbol for natural logarithms
p_i—internal pressure, psi.
p_r—pressure for plastic flow to radius r, psi.
p_{ei}—internal pressure elastic-breakdown, psi.
R—diameter ratio, c/a
R_1—diameter ratio, b/a
r—radius, in.
r_0—radius, in. (initial value)
u—radial deformation, in.
σ_h—hoop stress, psi.
$\bar{\sigma}$—equivalent stress, psi.
σ_i, σ_o—tensile yield strength of inner component and outer component of two-shell vessel, respectively, psi.
σ_r, σ_u, σ_y, σ_z—radial stress, ultimate tensile strength, tensile yield strength, and longitudinal stress, respectively, psi.
μ—Poisson's ratio
θ—relaxation strain, $\epsilon_h + \mu\epsilon_z$; in./in., angle
ϕ—σ_0/σ_i
$\bar{\epsilon}$—equivalent strain, in./in.
ϵ_1, ϵ_2, ϵ_3—principal strains, in./in.
ϵ_h, ϵ_t, ϵ_z—hoop strain, tensile strain, and longitudinal strain, respectively, in./in.

[1] Research Associate, Engineering Research Laboratory, E. I. du Pont de Nemours & Co., Inc., Wilmington, Del.

(σ')—prime mark denotes residual stress
$(\Delta\sigma)$—stress reduction due to reverse yielding
$(\ )_c$*—closed-end cylinder
$(\ )_e$—elastic zone
$(\ )_p$—plastic zone
$(\ )_{uf}$—zone of unrestricted plastic flow

1. INTRODUCTION

Some analyses of data concerning the behavior of heavy-wall prestressed cylinders under increasing static pressure up to bursting at room temperature are presented. Most of the experimental procedure and many of the data have already been presented (**2–6**).[2]

The present paper deals primarily with optimum design; this is considered from the point of view of maximum residual stress distribution and minimum material. The final practical design may not always be optimum with respect to size of vessel or distribution of stress, but optimum with respect to cost, safety, process conditions (including corrosion effects), and availability of both materials and technical information concerning the properties of the material. With regard to the latter point, it is shown for autofrettaged steel vessels that theory and practice are, at times, considerably at odds and that the material properties have a profound influence on the ability of a particular steel to accept and retain the optimum calculated stress distribution.

Dilation is also important since deformation is of major concern in many pressure vessels; most designs must be based on considerations of both a limiting deformation and a limiting stress; therefore the mere calculation of the yield and burst pressures as ultimate design criteria is grossly incomplete.

Thermal stresses and considerations of creep and stress-rupture have not been included here; these subjects are to be treated in another paper.

2. BASIS OF PRESTRESSING

The fundamental basis for prestressing is to utilize favorably distributed residual stresses rather than increased weight of material to resist service loads. In the autofrettage process (**7–9**) the amount of residual stress left at the bore of a cylinder on release of pressure is dependent on the pressure, material, and diameter ratio (O.D./I.D.). Full autofrettage is obtained when the theoretical maximum residual stresses are left at the bore; this is 100 per cent autofrettage. If only half the maximum possible residual stresses are left at the bore, the cylinder is 50 per cent autofrettaged. These percentage values are not to be confused with per cent overstrain of the cylinder wall. Depending on the cylinder, various degrees of overstrain are required to induce various amounts of autofrettage. Full overstrain of the cylinder wall

* Appropriate σ or ϵ terms go in the brackets.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

is called 100 per cent overstrain, whereas, if the plastic zone has penetrated only half of the wall thickness, the cylinder would be 50 per cent overstrained. It will be shown, based on these concepts, that in some instances 100 per cent autofrettage can be accomplished by only 10 per cent overstrain of the cylinder wall.

In many instances a low-cost unalloyed steel which is weak from the standpoint of strength of material can be used in a prestressed cylinder to accomplish the same result as an expensive high-strength alloy steel. Also, aside from mechanical prestressing, it is often possible to increase the strength of an otherwise low-strength material by suitable metallurgical treatment which gives rise to a "strain-aging" effect; this will be discussed later. Depending on the material, the yield strength can be increased by 5–50 per cent or more by a suitable strain-aging treatment. Also, both mechanical and metallurgical effects may combine to give improved performance of a vessel.

3. OPTIMUM RESIDUAL STRESS

The concept of an optimum residual stress system is relatively new. Actually, it is more closely associated with heavy-wall cylinders prestressed by means of purely elastic straining, like shrink-fit vessels, than with prestressed cylinders which depend both on elastic and plastic straining like autofrettaged and thermally shocked cylinders. All types of shrink-fit cylinders, including constructions having different materials for the shell and liner, can be designed so that under excessive loads the component parts fail simultaneously (**3**); this is optimum design for shrink-fit cylinders. Optimum autofrettage design has already been mentioned.

4. RADIAL EXPANSION OF A HEAVY-WALL CYLINDER

A complete stress analysis of autofrettage requires calculation of initial bore yielding (**2, 7**), overstrain through the wall (**7, 9, 11, 13**), bursting pressure (**1, 2, 5, 8, 9**), dilations (**1, 8, 9, 10**), and residual stresses (**2, 4, 7, 9, 11, 13**). The details of most of these analyses are fairly well known and will not be repeated here.

One part of the analysis does, however, bear repetition since it is not very well known but should be appreciated by anyone involved in autofrettage studies—that is the possible occurrence of reyielding on release of autofrettage pressure.

Reverse Yielding

The cylinder can be expanded by internal pressure until the entire wall is in the plastic range. This procedure, however, is not always necessary in order to arrive at the theoretical maximum degree of residual stress at the bore of the cylinder. The theoretical maximum

amount of residual stress that can be supported by the cylinder wall is determined by the von Mises criterion,

$$2\bar{\sigma}^2 = (\sigma_h - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_h)^2. \tag{1}$$

Since the radial residual stress at the bore is zero, the highest stress system that can be accommodated is

$$(\bar{\sigma})_c^2 = (\sigma_h')^2 + (\sigma_z')^2 - \sigma_h'\sigma_z' \tag{2}$$

for a closed-end cylinder, and

$$(\bar{\sigma})_o^2 = (\sigma_h')^2 \tag{3}$$

for an open-end cylinder. In other words, when the equivalent residual stress system exceeds the compressive yield strength of the material, yielding occurs on unloading as well as it did on original pressurizing. It is a simple matter to deduce from Eq. 2 that the limiting condition for no reverse yielding is a fully overstrained closed-end cylinder having a diameter ratio of 2.22. For the open-end cylinder (Eq. 3) this limiting ratio is 1.94. For diameter ratios greater than these, reverse yielding will occur on release of autofrettage pressure if the pressure is greater than twice the elastic-breakdown pressure for a closed-end cylinder, or 1.83 times the elastic-breakdown pressure for an open-end cylinder. The theoretical optimum autofrettage pressure is thus, for most cases, about double the initial yield pressure for cylinders having diameter ratios in excess of 2.0, and the overstrain pressure for cylinders of lesser diameter ratio. Since the conditions of both open and closed ends yield about the same results the following reverse-yield analysis due to Prager and Hodge (12) will be assumed to cover all cases. For the case of reverse yielding,

$$p_r = \frac{2\sigma_y}{\sqrt{3}}\left[1 - \left(\frac{b'}{c}\right)^2 + \ln\frac{b'}{a}\right] \tag{4}$$

where b' is the reverse-yield boundary. Then, for the elastic portion of the wall, the stress reductions due to reverse yielding are:[3]

$$(\Delta\sigma_h)_e = -\frac{2\sigma_y}{\sqrt{3}}\left(\frac{b'}{c}\right)^2\left[1 + \left(\frac{c}{r}\right)^2\right] \tag{5}$$

$$(\Delta\sigma_r)_e = -\frac{2\sigma_y}{\sqrt{3}}\left(\frac{b'}{c}\right)^2\left[1 - \left(\frac{c}{r}\right)^2\right] \tag{6}$$

$$(\Delta\sigma_z)_e = -\frac{2\sigma_y}{\sqrt{3}}\left(\frac{b'}{c}\right)^2. \tag{7}$$

[3] For the open-end cylinder any σ_z term would always be zero.

In the plastic portion of the cylinder wall,

$$(\Delta\sigma_h)_p = -\frac{2\sigma_y}{\sqrt{3}}\left[2\ln\frac{r}{b'} + 1 + \left(\frac{b'}{c}\right)^2\right] \tag{8}$$

$$(\Delta\sigma_r)_p = -\frac{2\sigma_y}{\sqrt{3}}\left[2\ln\frac{r}{b'} - 1 + \left(\frac{b'}{c}\right)^2\right] \tag{9}$$

$$(\Delta\sigma_z)_p = -\frac{2\sigma_y}{\sqrt{3}}\left[2\ln\frac{r}{b'} + \left(\frac{b'}{c}\right)^2\right]. \tag{10}$$

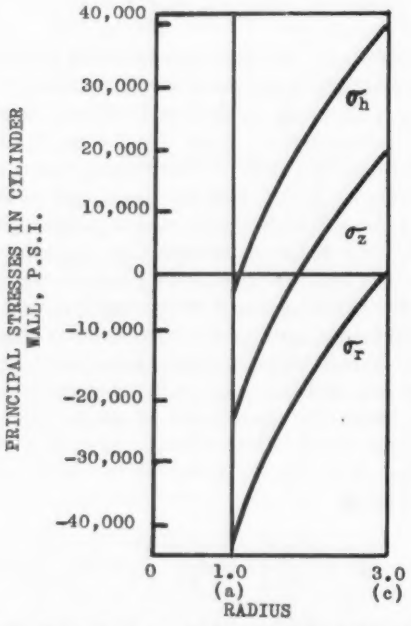


FIG. 1. Stress distribution in wall of closed-end cylinder at overstrain pressure (43,000 psi.).

The residual stress distribution is obtained by adding the stresses under pressure to the stress reduction given above, that is,

$$\sigma' = \sigma + \Delta\sigma. \tag{11}$$

In order to illustrate the reverse-yield effect, stress distribution plots have been made for a cylinder of diameter ratio 3.0 and having a tensile yield strength, σ_y, of about 35,000 psi. Since this cylinder has a diameter ratio greater than 2.22, reverse yielding can be induced by

pressurizing the cylinder at a pressure greater than twice the elastic-breakdown pressure. In this example, complete overstrain of the cylinder wall is assumed and the distributions so obtained are shown in Fig. 1. When the pressure is released, reverse yielding starts at a location defined by Eq. 4; the final residual stresses are calculated using Eqs. 5–11 and are plotted in Fig. 2. It can be seen that there is an increase in compressive stress that occurs as the distance from the bore increases. This condition does not necessarily mean that increased benefit is obtained by allowing reverse yielding to occur; this is shown

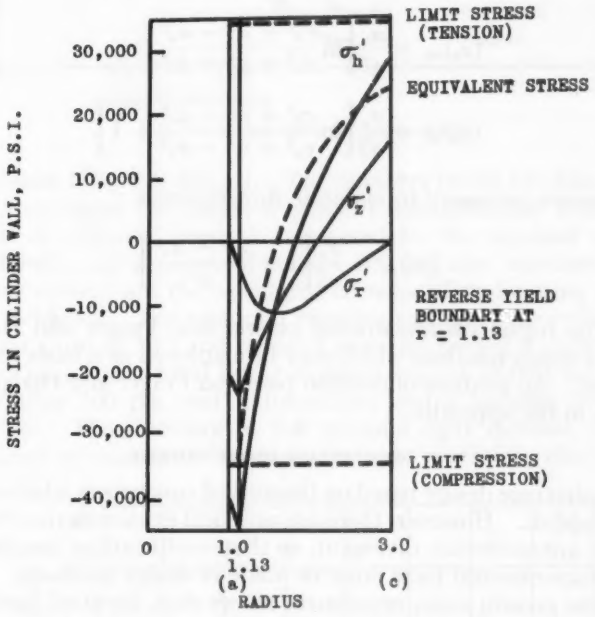


FIG. 2. Reverse yield residual stresses.

by the fact that the equivalent stress is constant through the reverse-yield zone. There may be some benefit in having a zone of high equivalent residual compressive stress but this has not been confirmed. Along with the possible benefit is a counter-effect; the equivalent tensile stress is increased at the outside surface of the cylinder and continues to increase as the reverse-yield boundary moves farther and farther from the bore. Thus, eventually, failure could presumably start at the outside of the cylinder, particularly under fluctuating stresses or in the presence of serious stress concentrators. The reverse-yield phenomenon will probably be most useful for autofrettaged cylinders in

which it is necessary to machine the bore following the autofrettage treatment.

In the region of unrestricted plastic flow following full overstrain (12), the initial bore and O.D. radii a_o and c_o increase to a and c with time. A particle which originally had a distance r_o from the axis of the cylinder is now under the following stresses when the bore radius a_o reaches the value a:

$$(\sigma_h)_{uf} = \frac{\sigma_y}{\sqrt{3}}\left[\ln\frac{r_0^2 + a^2 - a_o^2}{c_o^2 + a^2 - a_o^2} + 2\right] \tag{12}$$

$$(\sigma_r)_{uf} = \frac{\sigma_y}{\sqrt{3}}\left[\ln\frac{r_0^2 + a^2 - a_o^2}{c_o^2 + a^2 - a_o^2}\right] \tag{13}$$

$$(\sigma_z)_{uf} = \frac{\sigma_y}{\sqrt{3}}\left[\ln\frac{r_0^2 + a^2 - a_o^2}{c_o^2 + a^2 - a_o^2} + 1\right]. \tag{14}$$

The pressure necessary to maintain flow becomes

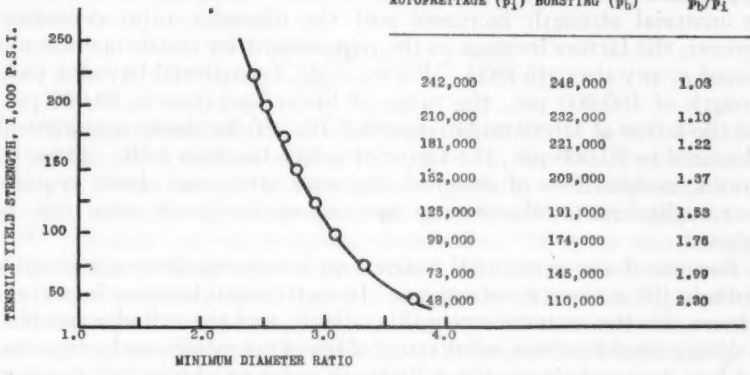
$$(p_i)_{uf} = \frac{\sigma_y}{\sqrt{3}}\left[1 + \frac{c_o^2 - a_o^2}{a^2}\right]. \tag{15}$$

In the region of unrestricted plastic flow Prager and Hodge (12) also give strain relations which may be employed as a basis to calculate dilations. An analysis of dilation based on Prager and Hodge's theory is given in the appendix.

5. AUTOFRETTAGE DESIGN CRITERIA

Autofrettage design based on theoretical concepts is relatively simple to accomplish. However, there are practical evidences that theory and practice are somewhat divergent, so that modifications consistent with known experimental facts must be made in design methods.

At the present time, experiment shows that, for steel, hardness and diameter ratio determine the ability of a cylinder to be satisfactorily autofrettaged. Details on a critical series of autofrettage experiments were presented by the author in an earlier paper (4) and will not be repeated here. The main results of those data have, however, been plotted in Fig. 3, along with some additional data acquired subsequent to the paper. Figure 3 shows, for various diameter ratios, the required yield strengths of the cylinder material, in order that a fully autofrettaged ($p_i = 2p_{yi}$) cylinder will operate linearly on subsequent pressure cycles to the autofrettage pressure. For the lowest strength material the minimum required diameter ratio is about 3.7, while for the strongest material the diameter ratio is about 2.5. This is a narrow range of diameter ratios, but within this range design can presumably be effected to almost any specification on elastic range operation and



INTERNAL PRESSURE, P.S.I. AUTOFRETTAGE (P_i)	BURSTING (P_b)	FACTOR OF SAFETY P_b/P_i
242,000	248,000	1.03
210,000	232,000	1.10
181,000	221,000	1.22
152,000	209,000	1.37
125,000	191,000	1.53
99,000	174,000	1.76
73,000	145,000	1.99
48,000	110,000	2.30

FIG. 3. Design curve for autofrettage.

burst pressure (Eq. 13, Ref. 5). The numbers in the left-hand column in Fig. 3 represent the range of elastic linear operation available for materials of different strength fabricated to the required minimum diameter ratio and pressurized to give 100 per cent autofrettage. In the center column are the calculated corresponding bursting pressures of these cylinders. For example, referring to Fig. 3, the cylinder with a diameter ratio of 3.75 and made of material having a yield strength of 50,000 psi., has a total linear elastic range of 48,000 psi. internal pressure after 100 per cent autofrettage, and a bursting pressure of 110,000 psi. The numbers to the extreme right indicate factors of safety based on maximum linear elastic range operation; that is, burst-

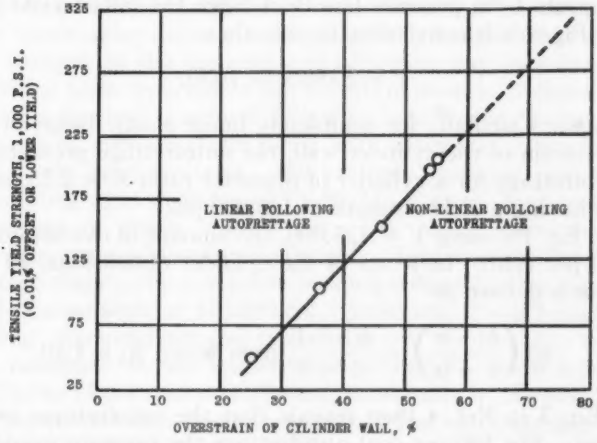


FIG. 4. Pressure-strain stability of autofrettaged cylinders.

ing pressure divided by autofrettage pressure. The factors decrease as the material strength increases and the diameter ratio decreases; however, the factors increase as the requirement for elastic operation is relaxed at any strength level. For example, for material having a yield strength of 100,000 psi., the range of linear operation is 99,000 psi., and the factor of safety on bursting is 1.76. If the elastic requirement is lowered to 50,000 psi., the factor of safety becomes 3.50. Thus, by various combinations of material, diameter ratio, and elastic requirement, cylinders to almost any pressure-strain specification can be designed.

Because of size or material restrictions it may not always be possible to obtain 100 per cent autofrettage. In such cases it becomes important to know, for the material strength available, and the cylinder specified by design considerations, what range of linear operation can be expected and how to autofrettage the cylinder in order to obtain this range of linear behavior. For autofrettage under these circumstances use is made of Fig. 4, which relates the permissible overstrain to the yield strength of the material. For example, as shown on Fig. 4, for a material having a yield strength of about 50,000 psi., 25 per cent overstrain is the maximum that can be tolerated by the cylinder in order for the cylinder to behave linearly on recycling to the autofrettage pressure. At the present time, Fig. 4 is applicable only to those cases where the autofrettage pressure is not high enough to cause reverse yielding on release of pressure. For example, using a material having a yield strength of 50,000 psi., the limiting overstrain is 25 per cent—this 25 per cent overstrain, however, must be caused by a pressure not exceeding that which will give 100 per cent autofrettage. This means that for cylinders of diameter ratio less than 2.22 it is necessary to cause the overstrain by a pressure less than twice the initial yield pressure. In using Fig. 4 it is convenient to note that

$$Y = 5350(x - 16.8). \tag{16}$$

Example: Calculate, for completely linear elastic behavior, the per cent overstrain of the cylinder wall, the autofrettage pressure and per cent autofrettage for a cylinder of diameter ratio $R = 2.25$, made of a material having a yield strength of 125,000 psi.

From Eq. 16, using $Y = 125{,}000$, the amount of overstrain is found to be 40 per cent. In terms of the cylinder dimensions, 40 per cent overstrain is defined as

$$40\left(\frac{b-a}{c-a}\right) = \frac{R_1 - 1}{R - 1}, \text{ from which } R_1 \text{ is } 1.50.$$

Use of Eq. 3 in Ref. 4 then reveals that the autofrettage pressure is 98,600 psi. For 100 per cent autofrettage the pressure would have to

be $2p_{yi}$ or 116,000 psi. Therefore, the amount of autofrettage accomplished is

$$\frac{p_r - p_{yi}}{p_{yi}} (100) = 70\%. \tag{17}$$

When performing the autofrettage treatment, it is well to keep in mind the effect of the end condition, that is, whether the cylinder has open or closed ends. In many cases, especially with large vessels, it is usually more convenient to autofrettage with a filler bar equipped with "O" ring seals, giving an open-end cylinder. In either case, any side holes in the cylinder are also prestressed, but allowance should be made for stress concentration; this is discussed in (6). If it is decided to autofrettage a closed-end cylinder, then especial care must be exercised since tangential holes in the cylinder wall (for valves) produce elliptical holes at the bore which give rise to high stress concentration effects at the end of the major axis. In order that the longitudinal stress in the cylinder does not induce an unduly high stress at an elliptical side hole the ellipse geometry has to be controlled within rather narrow limits. For example, for a cylinder of diameter ratio 2.5, the limiting ellipse axis ratio to avoid excessive stress and possible crack initiation is about 4 (see 6).

6. AUTOFRETTAGE STABILIZATION HEAT-TREATMENT

Autofrettage is frequently accompanied by a "stabilization" heat-treatment at around 600° to 700° F. The exact nature of what this treatment does is not known, but its effects can be at least two-fold. First, if residual stresses are present which are greater than the yield strength of the material at temperature, stress relief will occur. Under certain circumstances, apparently largely dependent on the steel composition, strain aging might also occur and this has the effect of increasing the strength of the material and offsetting any loss due to stress relief. Some tests reported in the literature seem to indicate that the benefits of a thermal treatment following autofrettage are of a metallurgical nature, such as strain aging with attendant increase in yield strength of the aged material. For example, data are reported (2, 4) for an autofrettaged cylinder of Cr-Ni-Mo-V steel, heat-treated to a yield strength of about 150,000 psi. The initial elastic-breakdown pressure was approximately 70,000 psi. Applying the principles of autofrettage theory, the maximum residual stress distribution would be achieved at a pressure of 140,000 psi. However, it was found that by "stabilizing" the autofrettaged cylinder at 650° F., linearity to 200,000 psi. was possible. It was known that at 650° F., stress relief to the extent of 10 to 15 per cent was effected; consequently, the only explanation for the remarkable elastic behavior of the autofrettaged cylinder following stabilization is that the yield strength of the cylinder material

was increased by some metallurgical process such as aging. Furthermore, the unloading curve from 200,000 psi. internal pressure was linear.

Since stabilization appears to have a marked effect on autofrettaged cylinders, it is desirable to select such a treatment with care so that optimum benefits will result. It appears that this post-autofrettage treatment can have at least three major effects: (*a*) partial relief of residual stresses imparted by autofrettage, (*b*) increase of the yield strength of the material by strain aging, and (*c*) reduction or elimination of mechanical hysteresis on recycling to the autofrettage pressure. The optimum stabilization temperature may thus be a compromise between the above effects since, for example, stress relief tends to lower and strain aging tends to increase the elastic-breakdown pressure following autofrettage and stabilization. In order to find the optimum stabilization temperature, if one exists, a method is proposed which utilizes the results of tests made on tension specimens; limited data indicate promise for the method. It is assumed that strain aging effects are primarily a function of the bore strain of a cylinder; consequently, tensile specimens are prestrained an amount equivalent to that at the bore using Eq. 25; thus,

$$\epsilon_t = \frac{u}{a\sqrt{3}} \tag{18}$$

where ϵ_t is the tensile strain equivalent to a cylinder bore diametral deformation u. Specimens thus prestrained are aged (say 1 hr. per inch of thickness) at various proposed stabilization temperatures, following which they are tested in tension at room temperature. The room temperature stress-strain curves then indicate any strain aging effect and under what conditions it is maximum. Companion tensile tests are also conducted at the proposed stabilization temperatures in order to indicate how the yield strength changes with temperature. With a decrease in yield strength, stress relief will occur if the residual bore stress exceeds the equivalent yield strength of the material at the stabilization temperature.

7. REDISTRIBUTION OF RESIDUAL STRESS ON MACHINING BORE OF PRESTRESSED CYLINDER

Cases frequently arise in connection with prestressed vessels where the bore has been subjected to erosion or mechanical wearing, thus necessitating refinishing; this usually means that the peak residual bore stresses are removed.

For example, an approximation of the amount of stress relief involved in machining the bore of a prestressed open-end cylinder can be obtained by considering the residual stress equations derived by Sachs and discussed in Ref. **2**:

$$\sigma_h' = \frac{E}{1-\mu^2}\left[(A_o - A)\frac{d\theta}{dA} - \frac{A_o + A}{2A}\theta\right] \tag{19}$$

$$\sigma_r' = \frac{E}{1-\mu^2}\left[\frac{A_o - A}{2A}\theta\right] \tag{20}$$

$$\sigma_z' = 0. \tag{21}$$

At the original bore-area value there are no relaxation strains, that is,

$$\epsilon_h = \epsilon_z = \theta = 0. \tag{22}$$

Therefore, at the bore, $\frac{d\theta}{dA}$ can be calculated and at the outside surface where $A_o = A$, θ can be calculated; these are terminal values. Various intermediate values can be obtained from Eq. 20, so that eventually a plot of θ *vs.* bore area is obtained, and new stress distributions obtained. Similar analyses can be made for closed-end cylinders.

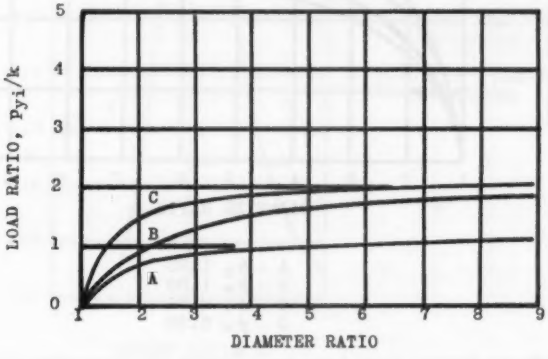


FIG. 5. Elastic-breakdown characteristics for three types of cylinders.

8. COMPARISON OF SHRINK-FIT AND AUTOFRETTAGE

A complete analysis of the relative merits of shrink-fitting and autofrettage is not possible here. Needless to say, the relative merits of each type of fabrication should be made in order to obtain optimum performance for the completed vessel. For purposes of discussion here some data reported for two-shell shrink-fit construction (3) will be used to compare with autofrettage. This analysis indicates some interesting possibilities and the methods, in general, could be applied to any other type of prestressing.

Insofar as elastic-breakdown pressure is concerned, consider Fig. 5, which shows the relation between diameter ratio and elastic-breakdown pressure for three types of vessels all constructed of identical materials. Curve *A* shows that for the initially stress-free cylinder little, if any gain can be expected in elastic-breakdown pressure by increasing the diameter ratio beyond 3.5 or 4.0. For autofrettage, Curve *C*, not much is gained by exceeding a diameter ratio of 5 or 6, whereas for two-shell shrink-fit cylinders, Curve *B*, the optimum pressure is not approached until a diameter ratio of over 10 is reached. Consequently, for a given application requiring a high elastic-breakdown pressure, it may be more advantageous to use a higher strength steel than to increase its diameter

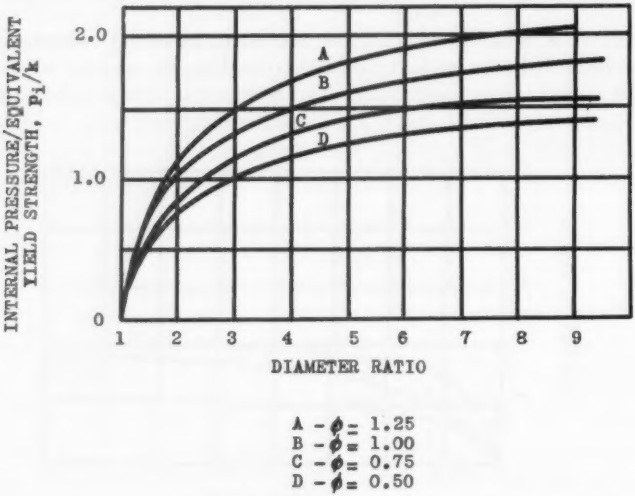


FIG. 6. Relation between elastic-breakdown pressure and diameter ratio for two-shell shrink-fit cylinder for variable shell and liner strength.

ratio. Also, comparing autofrettage and shrink-fit cylinders, note that when considering elastic-breakdown, autofrettage is superior to two-shell shrink-fit construction for all cases except infinite diameter ratio.

In Fig. 6 curves are shown which express the relation between elastic-breakdown pressure and diameter ratio for two-shell shrink-fit cylinders in which the yield strengths are different in the shell and the liner. The curves are plotted by using Eq. 54 of Ref. 3 in the form

$$\frac{p_i}{K} = 2\left[\frac{1+\phi}{2} - \frac{\sqrt{\phi}}{R}\right] \tag{23}$$

and plotting the left side of the equation as the ordinate. The practical significance here is that for a diameter ratio of 4, for example, the

elastic-breakdown pressure can be increased over a third by increasing the strength of the outside shell from 0.50 to 1.25 where σ_i is the strength of the liner.

An interesting aspect of two-shell shrink-fitting is the consideration of fabrication of a shrunk-wall cylinder such that it would have the same over-all diameter ratio and elastic-breakdown pressure as an autofrettaged cylinder; in this regard Fig. 7 may be used. This figure shows the variation with diameter ratio of the required ratio σ_o/σ_i for the shrink-fit cylinder. Two cases can be considered: *Case 1*—in which the yield strength of the material of the cylinder to be autofrettaged is the same as the yield strength of the outer shell of the shrink-fit cylinder,

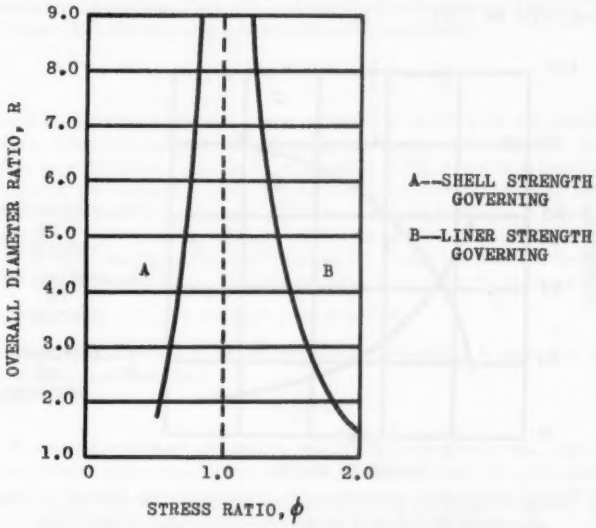


FIG. 7. Curves showing required ratio ϕ for shrink-fit vessel to give same elastic-breakdown pressure as autofrettaged cylinder of same diameter ratio.

(Curve *A*), and *Case 2*—in which the yield strength of the cylinder to be autofrettaged is the yield strength of the liner of the shrink-fit cylinder (Curve *B*). When the ratio ϕ is unity, both shell and liner of the shrink-fit cylinder have equal strength; in such a case shrink-fitting is equivalent to autofrettage only at infinite diameter ratio. When the ratio is less than unity the diameter ratio of the liner must increase in order to offset the fact that the shell has a lower yield strength; the reverse is true when the ratio ϕ is greater than unity. Consequently, noting Eq. 44 of Ref. (3), the limiting case for ϕ less than unity would be for the outside diameter of the liner to be equal to the outside diameter of the shell, that is, for b to equal c. In this case, the limiting value of ϕ would be $1/R^2$. For values of ϕ greater than unity the limiting case

would be for the outside and inside diameters of the liner to be equal; that is, for b equal to a. In this case, the limiting value of ϕ would be R^2. In Fig. 7 most of the important design cases where autofrettaged and shrink-fit cylinders of the same diameter ratio could be fabricated occur between $\phi = 0.50$ to 0.80 and between 1.25 and 2.00. For diameter ratios less than about 2.0, there is little possibility of combining a shell and liner to produce a shrink-fit cylinder equivalent to an autofrettaged cylinder in elastic-breakdown pressure, for cylinders of the same material. This is especially true for Case 2 since the liners would be so thin. For an autofrettaged cylinder of diameter ratio 4.0 and a yield strength σ_o for the shell and $1.49\sigma_o$ for the liner, $\sigma_o/\sigma_i = 0.67$. For this case, the liner would have a diameter ratio of 2.21 and the shell a diameter ratio of 1.81.

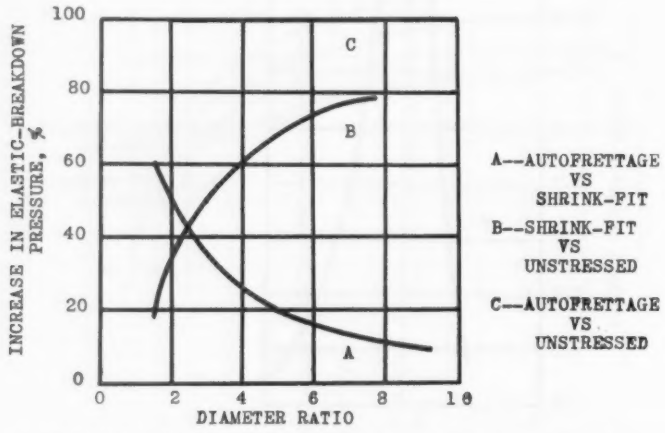


FIG. 8. Relative merits on elastic-breakdown pressure of unstressed, autofrettaged, and shrink-fit cylinders for optimum processing of each type.

As a final comparison between elastic-breakdown pressures for the three types of cylinders made of material having the same strength, consider Fig. 8. This figure indicates the per cent increase in elastic-breakdown pressure that can be expected over unprestressed cylinders for any diameter ratio. The comparisons are for theoretical optimums obtained for each type.[4] For the unprestressed cylinders, this optimum is given by Eq. 8, Ref. (5); for the autofrettaged cylinders, the optimum is $2p_{yi}$; for the shrink-fit cylinders the optimum is for equal diameter ratios of shell and liner and is given by Eq. 33, Ref. (3). Figure 8 shows, for example, for a diameter ratio of 5.0, that the elastic-breakdown pressure can be increased 100 per cent by using an autofrettaged cylinder rather than an unprestressed cylinder; an approximate 70 per

[4] For autofrettage, the material strength at R follows Fig. 3.

cent increase can be realized if a shrink-fit cylinder is used in preference to an unprestressed cylinder; 20 per cent increase can be expected in an autofrettaged cylinder as compared to a shrink-fit cylinder. In terms of design, this indicates that, theoretically, autofrettage is very attractive since, for the same factor of safety based on bursting, the range of elastic operation can be doubled. These remarks are true only for the two-shell shrink-fit construction which is commonly used. Considerable modification of the above is necessary when other shrink-fit conditions are used.

APPENDIX

DILATION OF CYLINDER IN REGION OF UNRESTRICTED PLASTIC FLOW

From conditions of compatability (13) and the theory of unrestricted plastic flow due to Prager and Hodge (12), the following relationship can be established

$$(\epsilon_h)_{af,\ r=a} = \frac{u}{r} = \frac{a - a_0}{a_0} = \left(\frac{a}{a_0} - 1\right). \tag{24}$$

Thus, bore hoop strains can be calculated (and corrected to the O.D. by the condition of constancy of volume if desired), and plotted against the pressure given by Eq. 15. One difficulty arises here with the value for σ_y. In Eq. 15, σ_y equals $\bar{\sigma}$ of Eq. 1, and it is implied that only the yield strength of the material controls flow and eventual fracture; this may be true if the pressure is held constant as in a creep test. But in monotonic loading it is believed that the ultimate strength, σ_u, also exerts its influence. One possible method of making a correction to σ_y in Eq. 15 is to recall the expression for "equivalent" strain, the abscissa for the ordinate, "equivalent" stress, given by Eq. 1. This strain value is given by the expression (11)

$$\bar{\epsilon} = [\tfrac{2}{3}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)]^{1/2}. \tag{25}$$

Assuming the longitudinal strain to be inconsequential and calling it zero, Eq. 25, because of constancy of volume, reduces to

$$\bar{\epsilon} = \frac{2}{\sqrt{3}}\,\epsilon_h. \tag{26}$$

Having Eq. 26, where $\bar{\epsilon}$ represents the tension strain, the proper value of σ_y can be deduced from an ordinary tensile stress-strain curve; for any particular value of ϵ_h, $\bar{\epsilon}$ is calculated and then $\bar{\sigma}$ is read directly from the stress-strain curve and this value is put into Eq. 15 as a replacement for σ_y. Fracture would presumably occur when $\bar{\sigma}$ equaled σ_u.

REFERENCES

(1) B. CROSSLAND AND J. A. BONES, *Inst. Mech. Engrs.*, (London), Vol. 172, p. 777 (1958).
(2) J. H. FAUPEL AND A. R. FURBECK, *Trans. ASME*, Vol. 75, p. 345 (1953).
(3) J. H. FAUPEL, *Machine Design*, Vol. 26, p. 114 (1954).
(4) J. H. FAUPEL, JOUR. FRANKLIN INST., Vol. 259, p. 405 (1955).
(5) J. H. FAUPEL, *Trans. ASME*, Vol. 78, p. 1031 (1956).
(6) J. H. FAUPEL AND D. B. HARRIS, *Ind. Eng. Chem.*, Vol. 49, p. 1979 (1957).
(7) R. HILL, "The Mathematical Theory of Plasticity," Oxford, The Clarendon Press, 1950.
(8) S. M. JORGENSEN, *Trans. ASME*, Vol. 80, p. 561 (1958).
(9) W. R. D. MANNING, *Engineering*, (London), Vol. 169, pp. 479, 509, 562 (1950).
(10) J. MARIN AND F. P. J. RIMROTT, Welding Research Council Report No. 41, 1958.
(11) A. NADAI, "The Theory of Flow and Fracture of Solids," New York, The McGraw-Hill Book Co., Inc., 1951.
(12) W. PRAGER AND P. G. HODGE, "Theory of Perfectly Plastic Solids," New York, John Wiley and Sons, Inc., 1951.
(13) S. TIMOSHENKO, "Strength of Materials—Vol. 2," New York, D. Van Nostrand Co., 1941.

THE FRANKLIN INSTITUTE

Minutes of the Stated Meeting

May 18, 1960

The Stated Meeting of The Franklin Institute was held at 8:15 P.M. in the Lecture Hall. Mr. Wynn Laurence LePage, President, called the meeting to order with approximately 210 members and guests present.

Mr. LePage stated that the minutes of the March 16, 1960, Stated Meeting had been published in the April issue of the JOURNAL and, there being no corrections or additions, these minutes were approved as published. The minutes of the April 20, 1960, Stated Meeting will be published in the next issue of the JOURNAL and will be presented for approval at the next Stated Meeting.

The President announced that the Board of Managers, at its meeting on April 20, 1960, had nominated the Franklin Medalist for 1960, Dr. Roger Adams, of the Department of Chemistry and Chemical Engineering, University of Illinois, for Honorary Membership in The Franklin Institute and, in accordance with our By-Laws, his name was presented to the membership for election. On motion duly made and seconded, Dr. Roger Adams was elected an Honorary Member of The Franklin Institute, his membership to be effective when the award is presented to him in October 1960.

The President recognized Mr. Morton Gibbons-Neff, Chairman of the Membership Committee, who introduced the members of The Institute who, in 1960, had completed fifty years of continuous membership. The President then presented a Gold Key and Gold Membership Card to Mr. Richard P. Brown, Dr. Joseph Samuel Hepburn, Mr. Joseph Wharton Lippincott, Mr. John D. Shattuck, and, *in absentia*, Mr. John Morris Weiss. The Secretary of The Institute was instructed to send Mr. Weiss his Gold Membership Card and Gold Key with our sincere regrets that he had been unable to attend the meeting.

Before introducing the speaker of the evening, the President announced that this meeting marked the occasion of the Third Annual William B. Coleman Lecture, which lecture had been established in 1956, in honor of a man whose long record of service to The Franklin Institute had been outstanding.

The President then introduced Dr. Morris Cohen, Professor of Physical Metallurgy, Massachusetts Institute of Technology, and the Francis J. Clamer Medalist in 1959. Dr. Cohen's subject was "Metallurgy and the New Science of Materials." In his talk, Dr. Cohen stressed recent work that is providing new and better materials for industry. His talk was well received by the members and guests. In thanking Dr. Cohen, Mr. LePage said it was hardly necessary to point out that we had been gently guided through intricate charts to make it seem easy for us to understand a most important subject.

The President adjourned the meeting at 9:40 P.M.

WILLIAM F. JACKSON, JR.
Secretary

HARRY AND LENORE WALTER MEMORIAL SCHOLARSHIP AWARDED

At the dinner preceding the Stated Meeting of The Franklin Institute on May 18, 1960, the Harry and Lenore Walter Memorial Scholarship was awarded to Richard Hill, top-ranking Senior in the Lansdowne-Aldan High School. Mr. LePage, President of the Institute, presented Richard with a letter as evidence of the award of $500, which will be repeated throughout Richard's undergraduate years, provided his scholastic record is satisfactory. The scholarship fund, held in trust by the Institute, under the terms of the will of Lenore Walter, is a memorial to herself and her late husband, Dr. Harry Walter. It is to be used ". . . in the education . . . of one or more boys deserving of such help."

BOOK NOTES

FLUIDIZATION AND FLUID-PARTICLE SYSTEMS, by Frederick A. Zanz and Donald F. Othmer. 513 pages, diagrams, 6 × 9 in. New York, Reinhold Publishing Corp., 1960. Price: $15.00 (trade edition); $12.50 (text edition).

This book provides a convenient, centralized information source on all major aspects of fluid-particle technology. The authors cover modern developments in fluid-solids processing, including the moving bed techniques, fluidization, and transport reactors. New unpublished material has been included in many of the chapters. The text covers the significance and methods of measurement of the angles of repose, internal friction, rupture, wall friction and slide. Details are reviewed of particle shape factors and their application in calculating pressure drop in fluid flow through particle beds. All available data on bubble velocity and bubble size in fluidized beds are included. For the research engineer, descriptions of pertinent phenomena in related fields abound.

MECHANICAL PROPERTIES OF INTERMETALLIC COMPOUNDS, edited by J. H. Westbrook. 435 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $9.50.

This is the proceedings of a special symposium held in Philadelphia in May 1959, sponsored by the Electrothermics and Metallurgy Division of The Electro-chemical Society, Inc. The seventeen chapters give broad coverage of the subjects which include: phenomenology of the mechanical behavior of intermetallics; theoretical and experimental investigations of particular properties; experimental techniques for the preparation and study of intermetallics; and an extensive review of the literature.

KINEMATICS, by Virgil Moring Faires. 468 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $8.50.

This extensive textbook presupposes a knowledge of elementary calculus and the rudiments of dynamics. The emphasis is approximately equally divided between the analytical aspects of motion, graphical solutions to common problems in mechanics, and practical applications to the design of actual machine components. Considerable attention is devoted to cams, gearing and fairly complex linkage chains, and a short chapter covers flexible connectors and friction wheels. An interesting chapter on synthesis of mechanics concludes the book. The more than eight hundred problems enhance the pedagogic value of the presentation. Many of the 140 references concern solutions of practical design problems.

THE SCIENTIFIC PAPERS OF G. I. TAYLOR, VOL. II, edited by G. K. Batchelor. 515 pages, diagrams, 6¾ × 10 in. New York, Cambridge University Press, 1960. Price, $14.50.

Devoted mostly to work on the mechanics of fluids, this second volume of the *Scientific Papers of G. I. Taylor* also contains his

writings on meteorology, oceanography and turbulent flow. The phenomenon of turbulence and its effects provide the main theme of the book, but a wide range of geophysical investigations is included. Diffusion in the atmosphere, formation of fog, tidal friction, tidal oscillations in gulfs, oscillations of the atmosphere, stability of stratified fluids and convection from sources have all engaged Taylor's attention at some time, and to each of these problems he has made a characteristically penetrating contribution. Some of the early papers on meteorology show clearly how Taylor gradually evolved the ideas which led to his well-known work in the 1930's on the statistical theory of turbulence. Meteorologists, oceanographers and geophysicists will find this work of precise interest; however, it will appeal to a wide range of scientists concerned with fluid mechanics in general and with turbulent flow in particular.

FLUID POWER CONTROL, edited by John F. Blackburn, Gerhard Reethof and J. Lowen Shearer. 710 pages, diagrams, 6 × 9 in. Cambridge (Mass.), The Technology Press; New York, John Wiley & Sons, Inc.; 1960. Price, $17.50.

This volume presents the fundamentals of the operation of power-control systems in which the working media may be either liquids or gases. Divided into four major parts, the book begins with a review of fluid properties and fluid mechanics. The second section deals in considerable depth with the theory and practice of hydraulic control components, with emphasis on control valves. The last two parts describe recent progress in the use of gaseous working fluids, especially high-pressure pneumatics with concepts of system analysis and design.

EXTRACTIVE METALLURGY, by Joseph Newton. 532 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $9.75.

Based on the author's "An Introduction to Metallurgy," second edition, this new and revised text covers the whole field of extractive metallurgy. The subject is presented through the unit process method, in preference to the metal-by-metal approach. Basic principles are discussed rather than detailed practices, which also include topics from physical metallurgy; metal crystals, equilibrium diagrams, and Gibbs' phase rule. The book was written primarily as a text for a first course in extractive metallurgy; as such it provides all the fundamental information required by the reader who is unfamiliar with the subject.

SURFACE EFFECTS ON SPACECRAFT MATERIALS, FIRST SYMPOSIUM, edited by Francis J. Clauss. 404 pages, diagrams, 8½ × 11 in. New York, John Wiley & Sons, Inc., 1960. Price, $11.50.

This volume includes the papers and discussions presented during the first symposium, held May 12–13, 1959 in Palo Alto, California, on the requirements of materials for temperature-control surfaces of spacecraft and the behavior of material surfaces in space. Among the subjects reviewed are the methods of calculating the required radiation characteristics of surfaces, experience with temperature-control systems on satellites and lunar probes launched to date. Data are also considered on the sublimation of material into ultrahigh vacuum, environmental testing for spacecraft, ultraviolet radiation in space and its effects on organic materials, interplanetary dust distribution and its erosion of surfaces, and atomic and molecular sputtering. The symposium was sponsored jointly by the U. S. Air Force Air Research and Development Command and the Lockheed Aircraft Corporation Missiles and Space Division.

ECONOMIC CONTROL OF INTERCONNECTED SYSTEMS, by Leon K. Kirchmayer. 207 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $12.50.

This book has as its primary thesis the treatment of mathematical methods, computers, and controllers to obtain the optimum economic operation of interconnected electric utility systems. It was written as a companion volume to the author's "Economic Operation of Power Systems" (Wiley 1958.) The theories featured in this new work represent pioneering applications of optimalizing computer control in a continuous process industry. The use of the power tool of Diakoptics conceived by G. Kron is also illustrated, providing a basic

philosophy for obtaining solutions of large systems from solutions of component parts of the total system.

FAST-NEUTRON SPECTROSCOPY, by B. V. Rybakov and V. A. Sidorov, edited by N. A. Vlasov. 121 pages, diagrams, 8½ × 10¾ in. New York, Consultants Bureau, Inc., 1960. Price, $17.50.

Translated from the Russian, this work is devoted to time-of-flight methods which represent not only a survey of work that has been done, but to a considerable extent, a description of the authors' own experiments and original results. As Soviet pioneers in the field, working with the one and a half meter cyclotron at the Institute for Atomic Energy, they have developed one of the first and possibly best of the presently available spectrometers. The electronic techniques used in fast-neutron spectroscopy (0.3–30 Mev) are described in detail. This first monograph is intended for scientists, engineers and technicians working in experimental physics, as well as for students in advanced courses.

GROWTH OF CRYSTALS, VOL. II, edited by A. V. Shubnikov and N. N. Sheftal'. 178 pages, diagrams, 8½ × 10¾ in. New York, Consultants Bureau, Inc., 1960. Price, $16.00.

This second volume, translated from the Russian, contains twenty-four papers on crystal growth, in addition to the late S. K. Popov's major contribution, in Volume I, on growing synthetic corundum. Some of the papers represent work that has been done in the USSR since the first conference; others break entirely fresh ground. The book is divided into three sections: Theoretical and experimental studies; Growing monocrystals, apparatus, methods and accessory operations; Reviews and discussion papers.

YOU CAN ADVANCE SCIENCE EDUCATION

Today, more than ever before in its 135-year history, there is vital need for The Franklin Institute to effectively promote education in science and technology. It is imperative that we meet this challenge by providing adequate educational opportunities in these fields. This requires vision, objective planning, and money. We have more than enough of the first two requisites, but far too little of the third.

Our programs are aimed at professional scientists and industry, as well as the lay public and young people seeking inspiration and guidance in choosing a career. The Institute's educational programs are impressive, for they begin with students in the early grades of our elementary schools and continue throughout an individual's professional or industrial life. With more funds at our disposal, the scope and vigor of these activities could be greatly increased.

The Franklin Institute is not richly endowed. It is a non-profit organization, depending for encouragement and support on an understanding public. Capable and conservative management assures wise administration of all funds.

Your gift or bequest, large or small, will be deeply appreciated and will be used effectively to broaden the Institute's educational usefulness. There is a warm satisfaction in giving financial support to an organization that has pioneered in, and is dedicated to, the advancement of science and technology through education.

When property is transferred, title should be in the name of The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts.

The Secretary of The Franklin Institute will gladly furnish you with additional information. Write to him at The Franklin Institute, Benjamin Franklin Parkway at Twentieth Street, Philadelphia 3, Pennsylvania.

PUBLICATIONS RECEIVED

CHEMICAL TECHNOLOGY OF PETROLEUM, by William A. Gruse and Donald R. Stevens. Third edition, 675 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $15.00.

COMPLEX VARIABLES AND APPLICATIONS, by Ruel V. Churchill. Second edition, 297 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $6.75.

CALCULATIONS OF ANALYTICAL CHEMISTRY, by Leicester F. Hamilton and Stephen G. Simpson. Sixth edition, 334 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $5.95.

DIFFERENTIAL EQUATIONS, by Ralph Palmer Agnew. Second edition, 485 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $7.50.

MODERN COLLEGE ALGEBRA, by Julian D. Mancill and Mario O. Gonzalez. 386 pages, diagrams, 6 × 9 in. Boston, Allyn and Bacon, Inc., 1960. Price, $6.25.

THE STAGES OF ECONOMIC GROWTH. A NON-COMMUNIST MANIFESTO, by W. W. Rostow. 179 pages, 5½ × 8 in. New York, Cambridge University Press, 1960. Price: $3.75 (cloth); $1.45 (paper).

MATHEMATICS IN ACTION, by O. G. Sutton. Reprint of book first published in 1954. 236 pages, diagrams, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.45 (paper).

THE PHILOSOPHY OF SCIENCE, by Stephen Toulmin. Reprint of book first published in 1953. 176 pages, diagrams, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.25 (paper).

FROM ATOMOS TO ATOM, by Andrew G. Van Melsen. Reprint of a book first published in 1952. 240 pages, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.45 (paper).

MOLECULES IN MOTION, by T. G. Cowling. Reprint of a book first published in 1950. 183 pages, diagrams, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.35 (paper).

SCIENTIFIC EXPLANATION, by R. B. Braithwaite. Reprint of a book first published in 1953. 374 pages, diagrams, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.85.

PHYSICS AND MICROPHYSICS, by Louis de Broglie. Reprint of a book first published in 1955. 286 pages, 5¼ × 8 in. New York, Harper & Brothers, 1960. Price, $1.50 (paper).

NATIONAL BUREAU OF STANDARDS NEWS

CONDUCTIVE FLOORING FOR HOSPITAL OPERATING ROOMS

Conductive flooring is as serviceable as nonconductive flooring of the same type, and it reduces explosion hazards, according to a recent National Bureau of Standards investigation[1] sponsored jointly by the Army, Navy, and Air Force. The Bureau also found that current methods for measuring the resistance of installed conductive floors reasonably simulate the conditions these floors meet in service. The investigation was carried out by T. Boone, F. Hermach, E. MacArthur, and R. McAuliff.

Although the flooring was investigated primarily for use in hospital operating rooms, many of the results obtained should apply equally well to floors that are used in other locations such as munition plants or storage depots for explosives. Consequently, architects may now choose conductive flooring materials for all these uses, within some limitations, on the basis of the behavior of similar nonconductive materials.

Explosive vapors are often present in hospital operating rooms. If static electricity is allowed to accumulate, it can cause sparks having sufficient energy to ignite these vapors.[2] The most effective means for reducing this hazard is to keep the electrical resistance between objects in the area so low that the minimum sparking voltage (about 400 v) is never attained. Conductive flooring provides the necessary linkage because most objects normally rest or move upon the floor.

Electrical resistance of flooring material is generally lowered by adding carbon black which is produced by thermally decomposing acetylene gas under carefully controlled conditions. Carbon black is dispersed in the mixes that are shipped for on-the-job composition of latex, concrete terrazzo, and the setting bed for the ceramic tile. On the other hand, the carbon black is dispersed during manufacture in ceramic, linoleum, rubber, and vinyl flooring.

Specifications and measurements of flooring resistance are usually made according to the method recommended by the National Fire Protection Association (NFPA). According to this,[3] resistance is measured by means of a 500-v. ohmmeter connected to 2 similar electrodes that simulate footwear and conductive rubber objects. A conductive floor should have a resistance low enough to prevent the build-up of dangerous static-charge voltages, yet its resistance should be high enough to prevent sparks or severe shocks from electric power lines or equipment that might become defective. According to NFPA, floors should have a resistance falling between 25,000 and 1,000,000 ohms when measured between specified electrodes that are 3 ft. apart. The

[1] For further technical details, see "Conductive Flooring for Hospital Operating Rooms," by Thomas H. Boone, Francis L. Hermach, Edgar H. MacArthur, and Rita C. McAuliff, *J. Research NBS*, Vol. 63C, No. 2 (1959).

[2] "Hazards from Static Electricity," *NBS Tech. News Bul.*, Vol. 36, p. 82 (June, 1952).

[3] "Code for Use of Flammable Anesthetics, (Safe Practice for Operating Rooms)," N.F.P.A. No. 56, National Fire Protection Association, 60 Batterymarch Street, Boston, Massachusetts.

upper limit specified by NFPA provides a safety factor of more than 10, even under conditions of low humidity, if materials such as wool and plastics are prohibited.

In the Bureau's laboratory investigation, however, electrodes were spaced at 1-ft. intervals—necessary because of the 18 × 18-in. sample size that was chosen for convenience in the experimental work. A preliminary study showed that this spacing would yield experimental results very close to those obtained from tests of an entire floor at an electrode spacing of 3 ft. Electrical tests were made under deliberately varied conditions of relative humidity and applied voltage. In addition to the laboratory tests on samples of flooring, field tests were made on 5 different types of conductive floors installed in hospitals in the Washington (D. C.) area.

Results from studies of the influence of moisture and other factors showed that conductive floors should give satisfactory service in hospital operating rooms if certain precautions are observed. If oxychloride floors are used, the humidity of the air in the room in which they are installed should be controlled, and the cleaning schedule for the floor should be carefully established and maintained. Laboratory tests indicate that otherwise the electrical resistance of the floor may fall outside the accepted limits.

The durability and appearance of at least two of the available materials (linoleum and rubber) may depend on periodic waxing. Conductive waxes containing carbon black which do not deposit an insulating film are available and should be esthetically as well as electrically satisfactory on these uniformly black floors. Sealers should probably not be used on conductive floors until proven satisfactory by extensive electrical tests.

To substantiate the electrical test results, additional tests were carried out in which each type of floor was actually used to reunite electrostatic charges. Thus, a direct test was made of each flooring's effectiveness in eliminating static electricity hazards.

There was excellent correlation between the electrical and electrostatic tests. For all but 2 of 12 specimens tested, peak voltage was less than 300 v. when the resistance between 2 standard electrodes was less than 1,000,000 ohms. Conversely, the resistance was greater than 1,000,000 ohms when the voltage exceeded 300 v. Because this voltage is below the minimum sparking voltage in air, the specified 1,000,000-ohm limit with the standard electrodes was proven to be a reasonably valid criterion of the performance of these floors.

The nonelectrical properties studied were indentation, scratch resistance, slipperiness, scrubbing, water absorption, and stain resistance. Results of each study showed that the characteristics of the conductive materials are comparable to those of the corresponding nonconductive materials.

Of the properties studied, scratch resistance is probably the most important in hospital operating room floors for the sake of cleanliness as well as for ensuring good contact resistance. Two methods were used for measuring scratch resistance. In the first, a diamond point was moved across a flooring sample and the scratch width was measured. Loads on the diamond point were 250, 500, 750, and 1000 grams. In the second method, a pedestrian traffic test ramp was used to simulate actual flooring use. Photographs were made after 11 months (100,000 passages) and compared with photographs of unexposed flooring to determine the extent of scratching, smudging, and other damage.

LENS DESIGN PROBLEMS SOLVED BY MATRIX ALGEBRA

The National Bureau of Standards has developed under the sponsorship of the U. S. Air Force an improved method that employs matrix algebra in designing optical lenses.[4] In this method, devised by O. N. Stavroudis of the Bureau's optical instruments laboratory, a system of partial differential equations defining optical image formation is written in the form of 4-by-4 matrices. The form of these matrices suggests a new method of designing lenses.

The greatest stumbling block to the lens designer is the difficulty of correlating the aberrations of an optical system with such parameters as the curvatures, separations, and thicknesses of the lenses and the indices of refraction. Thus the design of a lens is ordinarily a trial-and-error process, where values are assigned to the parameters, rays are traced, and the results are used to assign values representing the aberrations.

In the improved method, the aberrations of a lens can be defined in such a way that their relationship to the parameters is explicit. Here the elements of the 4-by-4 matrix represent the total aberrations of an optical system. If X is such a matrix, and the optical system represented by X has two components, then $X = X_1X_2$, where X_1 and X_2 are the X matrices for each of the components. Each X can be further split into additional components. This splitting can go on until the ultimate components are reached; these are (*a*) a refracting surface with which is associated a "refraction matrix X_R," and (*b*) the space between two refracting surfaces with which is associated a "transfer matrix X_T."

Most optical systems consist of spherical surfaces separated by homogeneous isotropic media (glass and air). In this case X_T depends on two parameters, the distance between two successive surfaces and the index of refraction of the medium between them, and X_R depends on the curvature of the surface and the index of refraction of the two media on either side of the surface. Thus the X matrix describing the total monochromatic geometric aberration of an optical system can be represented by a product of X_R and X_T matrices.

This principle can be used to set up a matrix equation whose solution is the design of a lens meeting certain prescribed requirements. Describing these requirements in terms of an X matrix and then setting it equal to a product of X_R-type and X_T-type matrices results in the required equation. Such an equation in 4-by-4 matrices is equivalent to a system of 16 simultaneous scalar equations. By assuming rotational symmetry and by making use of the system of differential equations mentioned above that apply to all optical systems, the complexity of the equations of condition may be reduced.

[4] For further technical information, see "Lens Design: A New Approach," by O. N. Stavroudis, *J. Res. NBS*, Vol. 63B, No. 1, p. 31 (July-September 1959). Part of this work was performed at the Imperial College of Science and Technology and included in a thesis submitted to the University of London.

CURRENT TOPICS

Double-Emitter Multi-Purpose Transistor.—A new developmental semiconductor device, called a double-emitter transistor, which engineers predict will simplify the designs of portable and automobile transistor radios, was described by two engineers of the Radio Corporation of America, at a technical session of the Institute of Radio Engineers Convention in New York.

This multi-purpose device, believed to be the first practicable transistor of its kind in the semiconductor industry, combines within one unit two functions (separate oscillator and mixer) which are normally accomplished by two transistors. The device makes it possible to obtain improved automatic-gain-control characteristics in transistorized broadcast-band receivers. Automatic-gain-control (agc) is a circuit arrangement which tends to keep the sound from a radio at a constant level when the strength of the received signal varies.

In conventional transistor radios, only limited automatic-gain-control can be applied to a conventional converter stage. The use of the double-emitter transistor makes it possible to apply agc directly to the mixer portion without affecting the oscillator portion.

Another technical advantage of the double-emitter transistor is freedom from oscillator blocking under transient strong signal conditions.

This multi-junction drift-field transistor has two alloyed *p*-type emitters, an *n*-type base, and one alloyed *p*-type collector. The two emitters are processed so that they can function independently of one another.

Technical feasibility of the device and its application as a combination oscillator-mixer capable of being gain-controlled have been demonstrated. Among the first significant results expected of this developmental transistor will be the design of broadcast-band receivers exhibiting excellent agc performance with only a frequency converter stage and a single intermediate-frequency amplifier stage without an overload or clamping diode.

World-Wide Transmission System. CODIT (*CO*mputer *DI*rect to *T*elegraph), an advanced fully automatic data transmission system with capabilities of transmitting reduced missile test information to any point in the world, was demonstrated at the Air Force Missile Test Center recently.

Now in operation to transmit predicted trajectory data from Cape Canaveral, launching site of ballistic missiles tested on the Atlantic Missile Range, to radars positioned along the 5,000 mile range, CODIT proved its capability of providing accurate reduced data to radars at strategic points on a world-wide range or to data analysis centers anywhere in the world.

Developed by RCA engineers at the Air Force Missile Test Center, CODIT utilizes presently operational systems in providing automatic data reduction and transmission at a speed heretofore not achieved.

The most important advantages of CODIT, says L. B. Haggadone, RCA project engineer, are automation and economy. Complete automation removes any possibility of human error,

while use of existing range and commercial communication systems provides low cost operation. By reducing the data prior to transmitting over a commercial system, the volume of data may be limited to that required, while needless volume would be sent over great distances if the entire data collection were transmitted.

In the recent Atlantic Missile Range demonstration, simulated ballistic missile tracking data were transmitted from Ascension Island, 5000 miles from Cape Canaveral, received by a CODIT receiver at the Cape and automatically reduced by a high speed computer. Again untouched by human hands, the reduced data were fed directly from the computer into a commercial land line teletype system that immediately displayed the reduced data at the Air Force Air Research and Development Command's National Space Surveillance Control Center, Cambridge, Mass.

The reduced data were read in Cambridge less than 30 seconds after the raw data were transmitted from Ascension Island.

By utilizing commercial land line, cable and radio networks, data might be collected anywhere in the world, reduced automatically and received at any other point in the world in a matter of seconds.

Removal of Atomic Waste from Water Supplies.—Municipal water plants, though not designed for the purpose, can remove a substantial fraction of radioisotopes from water they process, studies at the Hanford atomic laboratories, Richland, Wash., indicate.

Robert L. Junkins, a senior General Electric engineer at Hanford, reported to the Nuclear Congress on a year's sampling of the Pasco municipal water system filter plant 39 miles below Hanford nuclear reactors. He explained that General Electric, which operates Hanford for the Atomic Energy Commission, conducts this sampling as part of an extensive environmental monitoring program in Hanford's vicinity to provide guidance in the control of plant operation practices and to assure that the release of atomic wastes in the area is well within recognized limits.

Hanford's reactors use some Columbia river water for cooling purposes. Effluent from the reactors when discharged into the river contains several radioisotopes generated during its quick pass through the reactors. Columbia river water at Pasco contains about 15 per cent of the maximum concentration of radioisotopes permitted by national radiation health authorities. After treatment in the Pasco filter plant, he said, the 15 per cent of the permissible concentration is reduced to about 5, even though this plant was not designed and is not operated for this purpose.

He said the plant's efficiency in removal of radioisotopes varies with the seasonal demand for water. It is least efficient during summer months when the water plant is operating near capacity and there is less time for radioactive decay as the water passes through the treatment plant. However, this corresponds with the highest flow rate of the Columbia river when the effluent from the reactors is diluted most by increased water from the mountain snow melt.

Some of the radioisotopes are removed by sedimentation and filtration. The solid wastes from these sources are periodically removed by flushing. Because of radioactive decay, there is little accumulation of radioisotopes

in these solids or elsewhere in the plant.

U. S. Television Weather Satellite.—The United States' new TIROS satellite, carrying the nation's most advanced space-borne television "eye" to study the world's weather, is the information-gathering element in a complex satellite-and-ground system developed for the U. S. Government by the Radio Corporation of America.

The satellite comprises perhaps the most elaborate electronics package yet sent into orbit around the Earth, containing specially designed miniature television cameras, video tape recorders, transmitters, solar cell and rechargeable battery power supplies, and an array of control and communications equipment.

Speeding along its course in space, the satellite is linked to an extensive ground network of tracking and receiving stations, data-processing systems, and programming and control centers. Together, the satellite and ground equipment form a unified system to gather and analyze world-wide data on cloud formation in the Earth's atmosphere.

The TIROS system is an important new experimental tool for meteorologists. Its mission is the visual observation of cloud formations over large portions of the Earth to produce new information about such weather phenomena as hurricanes, typhoons, and the movement of weather fronts. Besides providing unprecedented and comprehensive visual coverage of global weather on a swift and continuous basis, the satellite will give meteorologists for the first time a means of checking the accuracy of present weather reporting from ground stations around the world. Thus it promises not only to lay the groundwork for new satellite weather services in the future, but also to increase the effectiveness of present world-wide weather information gathering methods.

The path of the satellite, circling the globe from west to east about every 90 minutes at an altitude of about 400 miles, will permit cloud observations throughout a belt extending from the latitude of Santa Cruz, Argentina, in the south, to the latitude of Montreal, Canada, in the north. During its planned operating lifetime of 90 days, the satellite is expected to complete about 1,300 orbits, covering every part of this belt many times.

Operation of the system involves the following sequence of three principal functions carried out by an intricate combination of electronic subsystems in the satellite and on the ground:

1. Preparation of a program and transmission of commands to the satellite at the start of each orbit, including instructions for picture-taking in specific areas of interest to the meteorologists.

2. Operation of the television and other equipment in the satellite during its passage, in response to the control commands received from the ground.

3. Read-out of the pictures and other information from the satellite on command from the ground at the completion of each orbit.

The principal ground stations are situated at Kaena Point, Hawaii, and Fort Monmouth, N. J. During all but three or four orbits each day, the satellite passes within range of one of the stations, permitting readout of information and programming of the satellite electronic apparatus for the next passage. In addition, two "backup" stations are situated at the Space Center of RCA's Astro-Elec-

tronic Products Division at Princeton, and at Cape Canaveral, Florida.

The satellite itself incorporates advanced concepts in space communications and the remote control of satellite functions. Its development required not only the design of new miniaturized television equipment and control devices, but also special methods of controlling the satellite's rate of spin, its attitude in relation to the Earth's surface, and its thermal balance during the 90-day period. Shaped like a drum 42 in. in diameter and 19 in. in depth, the vehicle is covered on its top and sides with over 9000 solar cells which generate electricity directly from sunlight to charge an assembly of small storage batteries that power the electronic equipment.

The key function of the satellite is performed by two complete television systems capable of taking a series of still pictures of cloud formations during each orbit. Each employs a specially-developed RCA ½-in. Vidicon camera small enough to hold in the palm of a hand. One of the cameras is equipped with a wide-angle lens for viewing clouds in an area about 800 miles on each side. The other employs a narrow-angle lens to scan cloud details in a smaller area.

Linked to each camera is a miniature television magnetic tape recorder specially designed by RCA engineers for satellite use. Each of the recorders stores up to 32 individual pictures taken by the camera during a single orbit. When the satellite passes within range of one of the ground stations, a command signal causes the information to be read from the tape into the satellite transmitter for transmission to the earth. For picture-taking while the satellite is within range of the station, the cameras can be instructed to feed their information directly to the transmitter rather than to the tape storage systems.

To permit meteorologists to identify the direction of each picture, the satellite carries a "north indicator" system which automatically measures the angle of the sun in relation to the vehicle. The resulting signal, sent to the ground with each picture, is processed by a "sun angle computer" and displayed on a television screen beside the picture to indicate the north direction of the image. In addition, an infra-red detector on the satellite senses the crossing of the earth's horizons as the vehicle rotates at its spin rate of 12 revolutions per minute. This information also is transmitted to the ground and is processed to determine the attitude in space of the vehicle at the time of picture taking.

At the start of each orbit, the television cameras can be instructed electronically to photograph an area of specific interest—such as a typhoon center over the Pacific or a hurricane in mid-Atlantic. The instructions, prepared at the NASA Computing Center in Washington in cooperation with specialists of the United States Weather Bureau, are sent to the ground stations. At the appropriate station, the program is sent in the form of radio signals to an "electronic clock" in the satellite. The clock stores the instructions somewhat in the fashion of a remotely operated alarm clock, causing the cameras to start a sequence of operations at the specified time during the succeeding orbit, as the satellite passes over the region of particular interest.

As the satellite swings around the Earth and comes again within range of a ground station, a command signal is sent from the ground for transmission of the cloud cover pictures stored on the tape. At the ground station, the information is displayed

on a television picture tube and recorded on another magnetic tape system. The image on the picture tube also is photographed for further study and future reference by experts of the Weather Bureau and other services.

Mr. Sternberg pointed out that the ground command and control system has been designed by RCA engineers to permit either automatic or manual operation of the satellite system, or a combination of both. With automatic operation, the program for the orbit is pre-set and each step is started by the electronic clock control in the satellite. With manual operation, the commands are transmitted by an array of push-buttons at the ground station, permitting direct operation of the television cameras while the satellite is within range, and providing an opportunity for checking individual satellite functions. By combining the two methods, the program is pre-set, but each sequence is started manually by push-buttons at the ground station when the satellite is within range.

Sun as Relay for Radio Signals.—Possible use of the sun as a huge reflector to relay radio signals between distant points on the Earth during periods when the moon is not available was described by an RCA scientist at the international convention of the Institute of Radio Engineers.

Donald J. Blattner, of the technical staff of RCA Laboratories at Princeton, N. J., told the radio engineers that such a method is technically feasible with presently known transmission techniques, and may be justified by the growing demand for more communication channels.

Recalling recent successful experiments in bouncing radar pulses from the sun, Mr. Blattner said that the solar atmosphere of ionized gases also can reflect coded radio signals with sufficient reliability for relaying limited amounts of data. He added that successful accomplishment would require a system including large antennas, powerful transmitters, and advanced electronic data-handling equipment.

According to Mr. Blattner, signals transmitted at a frequency of 40 megacycles would be reflected from the corona which forms the outermost portion of the sun's atmosphere. He said that such a system would require a transmitter with a power output of about one million watts, and a 120-ft. parabolic antenna capable of following the sun across the sky. Because of the rotational speed of the sun, he added, there would be some "smearing" of the relayed signals, necessitating a wide-band receiver, with which would be associated data-handling equipment to process the incoming information.

To overcome the spurious noise created by the solar atmosphere, Mr. Blattner suggested the use of 20-second pulses for communicating coded information. He pointed out that this would necessarily limit the speed of communication, since a single digit would require several minutes for transmission, while a complete word would take nearly half an hour. In addition, the signal would take approximately 16 minutes to cover the round-trip distance between the earth and the sun, traveling at the speed of light.

While such a system would be relatively slow and limited in its capacity, he added, it would nevertheless provide an alternate means of relaying useful amounts of coded data at times when the moon is not available for relaying.

Advanced Oscilloscope.—What is believed to be the world's fastest, most advanced oscilloscope was publicly shown for the first time at the recent annual Institute of Radio Engineers Show. The scope, Model 707 and 708 in the Milli-Mike line of high speed oscilloscopes manufactured by the Boston scientific firm of Edgerton, Germeshausen & Grier, Inc., is the result of over a decade of design, development and manufacturing experience in millimicrosecond and sub-millimicrosecond systems and instrumentation. The new Milli-Mike can detect and display phenomena lasting only 20 billionths of a second.

The Model 707 and 708 Milli-Mike is a traveling wave deflection-type oscilloscope designed to display both single transient and repetitive phenomena in the millimicrosecond and sub-millimicrosecond regions. The new scope emphasizes simplicity, ease of operation and versatility. Typical applications are measurements of high speed semiconductor diodes and the rise and fall times of high speed switching transistors. Other applications include high frequency measurements, jitter measurements, single transient investigation, nuclear research, photomultiplier output display and coaxial discontinuity measurement.

The Model 707 and 708 Milli-Mike oscilloscope has an extremely high frequency response. The bandwidth is rated as d.c. to 2000 megacycles, usable to 3000 megacycles. Rise time is 0.2 millimicrosecond. For greater flexibility, two different signals can be applied at the same time. Six calibrated sweep speeds are available for operation ranging from 5 millimicroseconds to 3000 millimicroseconds in the unmagnified condition.

The display area may be varied from 1 × 1.5 cm. at 20 KV to approximately 3 times this size at 3 KV permitting more information to be contained in the unmagnified display than is normally found in a conventional 5-in. KRT oscilloscope.

Two metal ribbon helices with a characteristic impedance of 100 ohms comprise the upper and lower vertical deflection structure in the TW cathode ray tube. The signal is applied directly to the input of one or both helices by means of a transmission line. No signal amplification is used; thus bandwidth limitation, noise generation and signal distortion are minimized.

All controls are located on the front panel for easy operation and are of the type familiar to oscilloscope users. A magnifier control adjusts the size of the display image.

A high quality optical reticle attaches to the front of the KRT. It is illuminated and a control is provided to vary the amount of illumination. Photographs can be taken through the reticle with any standard camera so that a record containing both the signal and the reticle may be obtained.

The Milli-Mike is available in two models, the 707 and 708. The 707 has a sensibility of 55 millivolts per trace width and a spot size of 0.002 in. The 708 has a sensibility of 5.5 millivolts per trace width and a spot size of 0.0015 in.

Other features include high writing speed, highest sensibility, variable display magnification, independent 100 ohm signal inputs and a balanced 100 ohm push-pull signal input, lockout, variable trigger delay and an integral coaxial delay line.

Radio System for Warning Drivers.—A portable transistorized roadside transmitter and car receiver system for alerting drivers to road conditions ahead has been developed at the

General Motors Research Laboratories, Warren, Mich. The roadside-to-vehicle communications system offers great advantages as a supplement to existing road signs. Passing motorists would be alerted to road condition, detours, exits, and emergency situations directly ahead by means of small low-power transmitters stationed at appropriate points along the road.

Permanent transmitter installations such as to provide information on the distance to the next freeway exit can be buried in bunkers. Emergency installations can be made within minutes by a policeman to detour traffic around an accident area, for example.

The receiver in the car will automatically receive every message through which the car passes and deliver it through the speaker. If the standard broadcast receiver happens to be on, its sound will automatically be cut off for the duration of the message so that the safety message can be heard, then returned after the message field is passed. It is intended that the highway alert radio be eventually made part of the standard car radio.

The system is unique in that the communication is effected through use of a magnetic induction field rather than a radiation field such as is used by radio and TV stations to get extensive coverage. A low power induction field is purposely used to restrict the communications, so that only those people directly concerned will receive the message. The operating frequency suggested is in the 10 to 14 kilocycle band, much lower than standard radio broadcasts.

Moderate-Price Computer.—A computer that is expected to make high-speed electronic data processing practical for more than 6000 of the 10,000 top companies in the U. S. has been introduced by Datamatic Division of Minneapolis-Honeywell. Known as the Honeywell 400, it is rated as able to perform 6000 operations per second.

Walter W. Finke, Datamatic president, described the "new-generation" computer as the "most powerful available in the moderate-price field." He said it will make high-speed electronic data processing economical for thousands of additional companies "because it provides more data processing per dollar spent than any previous equipment of similar size. Heretofore, high-speed electronic data processing has been practical for only a relatively small number of the largest corporations."

The Honeywell 400's capability of handling up to 6000 additions or subtractions per second is up to 10 times as fast as existing models in its price range. The new system also will have the fastest printing and sorting speeds and will use the industry's fastest and most reliable magnetic tapes, according to the manufacturer.

The new system will be compatible with Datamatic's larger Honeywell 800; thus a growing company can expand its data processing operations from the Honeywell 400 to the Honeywell 800 without difficulty.

The basic Honeywell 400 model—with a powerful central processor, four high-speed magnetic tape units, a high-speed printer and a card reader—will be available beginning in the summer of 1961.

The Honeywell 400 is designed for a broad range of business applications, although its unusual performance in the areas of storage and high-speed manipulation of large volumes of data make it exceptionally efficient at sorting and maintaining files, Finke explained.

The Honeywell 400 and Honeywell 800 will use identical tape systems, printers and card readers. Information compatibility of the Honeywell 400 and Honeywell 800 tapes also will be complete, which means that information recorded on one system may be read by the other system.

A programming language will be provided which can be translated by either a Honeywell 400 or Honeywell 800, and programs written in this language may be run on either system without modification. As a result, programs written for the Honeywell 400 may be checked out on the Honeywell 800 prior to the first deliveries of the smaller system.

The central processor of the Honeywell 400 will function at an internal speed of up to 6000 operations, such as additions or subtractions, per second. The core memory capacity will be 1,024 words of 48 bits each. All information passing through the central processor will be checked internally as it is read from or written on the magnetic tape. The system will be capable of performing both reading and writing operations simultaneously. High-speed printing can be performed simultaneously with any other operation.

The transfer rate of information to or from the high-speed magnetic tapes will be 64,000 characters or 96,000 decimal digits per second. Orthotronic Control, a unique feature on Honeywell data processing systems, will automatically detect and correct any errors in or damage to information recorded on the magnetic tapes.

The high-speed printer will operate at 900 lines per minute, while the card reader will handle 650 cards per minute.

In addition to the standard system of a central processor, four tape drives, a printer and a card reader, one or two additional tape mechanisms can be added at any time. One standard or high-speed card punch can be added to the system as optional equipment.

Electronic Survey System.—An electronic survey system that may provide rapid, accurate means of measuring distances and establishing positions is being evaluated by topographic engineers at the U. S. Army Engineer Research and Development Laboratories, Fort Belvoir, Va.

Called "Lorac," the system can be used to measure inter-island and other geodetic ties where line-of-sight conditions do not exist, and for establishing positions of boats, aircraft and land vehicles.

Utilizing a continuous-wave unmodulated radio transmission sent out from each end of the line to be measured, the system can measure distances up to 100 miles over mountains with minimum accuracies of 1:10,000. Over salt water, the range is increased to 200 miles and the minimum accuracy to 1:45,000.

To establish positions, the system employs two baselines extended from a central transmitting station and a transmitter at each end of the two baselines.

Both measurement and positioning employ aircraft mounted receiving stations.

Lorac was designed and developed by the Lorac Service Corporation, Tulsa, Okla. Under contract with the Laboratories, Lorac personnel tested the system according to a test plan approved by the Army.

INDEX TO VOLUME 269

January–June, 1960

SUBJECT INDEX

Entries with (C) and Current Topics; those with (N) are National Bureau of Standards News. Papers are listed under subject headings, with authors' names in parentheses following the title of the paper. Authors are listed, with titles of their papers in alphabetical order in the Author Index.

A

Aberrations. Lens design problems solved by matrix algebra, 497(N)

A.C. bridges. The accuracy of impedance measurements in a.c. bridges (Wolff), 299

Air conditioning. Sunshine powers air conditioners 160(C)

Airborne crane-shovel, 73(C)

Alloys. Yttrium as an alloying metal, 71(C)

Alumina powder used for potting, 160(C)

Amplifiers. A note on the Manley-Rowe energy relations for parametric amplifiers (Callen), 93

Anniversaries. Proceedings of the Stated Meeting held November 18, 1959. Presentation of the Vermilye Medal and observance of the 135th anniversary of The Franklin Institute, 125

Annual report of the Board of Managers for the year 1959, 315

Antennas. Design of corner reflector antennas, 437(N)

Anti-freezes. Non-corrosive anti-freeze, 353(C)

Aperture phase distributions. Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184

Applied mathematics. On Helmholtz's theorem in multiply-bounded and multiply-connected regions (Van Bladel), 445

Aslakson, Carl I. Boyden Premium, 146

Astronautics. On guidance and control requirements in astronautics (Roberson), 196

Atomic energy. Transistorized fuel gage, 253(C)

Atomic science. The science of yesterday, today and tomorrow (Swann), 163

Atomic theory. A simple unified field equation and hypothesis (Gresky), 105

Autofrettage. Some considerations of the mechanics and design limitations of autofrettage (Faupel), 474

Automatic control. Dynamic response of large complex systems (Mesarović), 274

Automation. Letter-sorting machine, 73(C)

B

Bartol Research Foundation

Balloon flight investigations of primary cosmic rays during solar disturbances (Pomerantz, Agarwal and Potnis), 235

Book reviews

Annals of the International Geophysical Year. Vol. I, 68

Averbach, B. L., D. K. Felbeck, G. T. Hahn and D. A. Thomas, ed.: Fracture, 246

Batchelor, G. K., ed.: The scientific papers of G. I. Taylor, Vol. II, 492

Battista, O. A.: The challenge of chemistry, 246

Bean, Richard L., Nicholas Chackan, Jr., Harold R. Moore and Edward C. Wentz: Transformers for the electric power industry, 151

Benson, Sidney W.: The foundations of chemical kinetics, 248

Book reviews (cont.)
Blackburn, John F., Gerhard Reethof and J. Lowen Shearer, ed.: Fluid power control 492
Blais, John F.: Amino resins, 153
Blumenstock, David I.: The ocean of air, 245
Borden, Perry A.: Telemetering systems, 246
Born, Max and Emil Wolf: Principles of optics, 68
Bracewell, Ronald N., ed.: Paris symposium on radio astronomy, 150
Brenner, Egon and Mansour Javid: Analysis of electric circuits, 152
Broehl, Wayne G.: Precision Valley: the machine tool companies of Springfield, Vermont, 70
Brown, Sanborn C.: Basic data of plasma physics, 245
Cable, J. Wesley: Vacuum processing in metalworking, 426
Carroll, Robert L.: The aerodynamics of powered flight, 426
Clauss, Francis J., ed.: Surface effects on spacecraft materials, first symposium, 493
Cleaveland, Frederic N.: Science & state government, 151
Corcoran, William H. and William N. Lacey: Introduction to chemical engineering problems, 247
Cram, Donald J. and George S. Hammond: Organic chemistry, 152
Dancy, T. E. and E. L. Robinson, ed.: Flat rolled products: rolling and treatment, 69
Dawson, J. K. and G. Long: Chemistry of nuclear power, 247
Dummer, G. W. A.: Modern electronic components, 69
Evans, W. H.: Experiments in electronics, 248
Faires, Virgil Moring: Kinematics, 491
Flaschka, H.: EDTA titrations, 347
Frenkiel, F. N. and P. A. Sheppard, ed.: Atmospheric diffusion and air pollution, 248
Gatlin, Carl: Petroleum engineering: drilling and well completions, 427
Gilmont, Roger: Thermodynamic principles for chemical engineers, 347
Gutenberg, Beno: Physics of the earth's interior, 346
Hausner, Henry H. and James F. Schumar, ed.: Nuclear fuel elements, 347
Hawley, George F.: Automating the manufacturing process, 152
Hemmerling, Edwin M.: Mathematical analysis, 68
Hinze, J. O.: Turbulence, 347
Hofmann, Joseph Ehrenfried: Classical mathematics—a concise history of the classical era in mathematics, 348
Horn, Milton B.: Acrylic resins, 427
Howard, N. E.: Standard handbook for telescope making, 149
Hunt, K. H.: Mechanisms and motion, 424
Jacobs, Horace, ed.: Advances in astronautical sciences, Vol. 4, 67
Jacobs, J. A., R. D. Russell and J. Tuzo Wilson: Physics and geology, 246
Jastrzebski, Zbigniew D.: Nature and properties of engineering materials, 247
Kiepenheuer, Karl: The sun, 153
King, Cuchlaine A. M.: Beaches and coasts, 347
Kirchmayer, Leon K.: Economic control of interconnected systems, 492
Kit, Boris and Douglas E. Evered: Rocket propellant handbook, 426
Klemperer, O.: Electron physics, 68
Kunin, Robert: Elements of ion exchange, 426
Laitinen, Herbert A.: Chemical analysis, 248
Landau, L. D. and E. M. Lifshitz: Theory of elasticity, 247
Marton, L., ed.: Advances in electronics and electron physics, 152
Matheson, J. A. L.: Hyperstatic structures, 64
Mathewson, C. H., ed.: Zinc, 66
Mendelssohn, K., ed.: Progress in cryogenics, 63
Morton, Maurice, ed.: Introduction to rubber technology, 347
Neugebauer, C. A., J. B. Newkirk and D. A. Vermilyea, ed.: Structure and properties of thin films, 248
Newton, Joseph: Extractive metallurgy, 492

Book reviews (cont.)
Novozhilov, V. V.: The theory of thin shells, 345
Overby, Charles A.: Aircraft and missile design and maintenance handbook, 426
Overman, Ralph T. and Herbert M. Clark: Radioisotope techniques, 348
Plastics engineering handbook, 348
Press, J. J., ed.: Man-made textile encyclopedia, 246
Resnick, Robert and David Halliday: Physics: for students of science and engineering, 427
Reza, Fazlollah M. and Samuel Seely: Modern network analysis, 152
Richards, Paul I.: Manual of mathematical physics, 247
Rouse, Hunter, ed.: Advanced mechanics of fluids, 66
Rybakov, B. V. and V. A. Sidorov (edited by N. A. Vlasov): Fast-neutron spectroscopy, 493
Schaaf, Samuel A. and Lawrence Talbot: Handbook of supersonic aerodynamics. Section 16, mechanics of rarefied gases, 346
Schwarzkopf, Paul and Richard Kieffer: Cemented carbides, 348
Segrè, E. and L. I. Schiff, ed.: Annual review of nuclear science, 153
Shames, Irving H.: Engineering mechanics: statics, 152
Shubnikov, A. V. and N. N. Sheftal', ed.: Growth of crystals, Vol. II, 493
Shvartz, G. L. and M. M. Kristal: Corrosion of chemical apparatus, 65
Slater, Noel B.: Theory of unimolecular reactions, 349
Smit, J. and H. P. J. Wijn: Ferrites, 69
Stallo, J. B. (edited by P. W. Bridgman): The concepts and theories of modern physics, 348
Taylor, Howard F., Merton C. Flemings and John Wulff: Foundry engineering, 69
Thermodynamic and transport properties of gases, liquids and solids, 149
Tucker, A. W. and R. D. Luce, ed.: Contributions to the theory of games, Vol. IV, 346
Turnbull, H. W., ed.: The correspondence of Isaac Newton, Vol. I, 1661–1675, 151
van der Plaats, G. J.: Medical x-ray technique, 247
Van Vlack, Lawrence H.: Elements of materials science, 151
Veinott, Cyril G.: Theory and design of small induction motors, 152
Wehr, M. Russell and James A. Richards, Jr.: Physics of the atom, 346
Westbrook, J. H., ed.: Mechanical properties of intermetallic compounds, 492
White, David C. and Herbert H. Woodson: Electromechanical energy conversion, 422
Wiberg, Kenneth B.: Laboratory technique in organic chemistry, 247
Wilson, A. C.: Industrial thermal insulation, 69
Woolf, Harry: The transits of Venus, 65
Zanz, Frederick A. and Donald F. Othmer: Fluidization and fluid-particle systems, 491
Boron. Zone melting of boron, 75 (C)
Boyden Premium. Carl I. Aslakson, 146
Business machines. Letter-sorting machine, 73(C)
Solid state data processing system, 432(C)

C

Cables. Undersea Arctic cable, 434(C)
Calibration of aircraft compass transmitters, 253(C)
Celestial mechanics. Satellite perturbations from extra-terrestrial gravitation and radiation pressure (Geyling), 375
Chemistry. The thunderstorm as a chemical phenomenon (Konow), 439
Circuit analysis. A note on the Manley-Rowe energy relations for parametric amplifiers (Callen), 93
Oscillating systems with one variable element (Cunningham), 81
Classical physics. Satellite perturbations from extra-terrestrial gravitation and radiation pressure (Geyling), 375
Coltman, John W., Jr. Longstreth Medal, 57
Committee on Science and the Arts
Abstracts of meetings: December 9, 1959, 57; January 13, 1960, 146; February 10, 1960, 234; March 9, 1960, 344;
Awards (*See* Boyden Premium, Cresson Medal, Longstreth Medal in main index for names of recipients. Individuals are listed alphabetically in main index, followed by name of medal, with page reference to the respective citation)

Communications. "Magic paper" for guarding secrecy of facsimile mail, 80(C)
Microwave data processing, 75(C)
"Varactor" diode used in radio receiver, 157(C)
Undersea Arctic cable, 434(C)
Computers. Moderate-price computer, 504 (C)
Solid state data processing system, 432(C)
Tiny new electron tube, 354(C)
"Twistors" for electronic memory devices, 256(C)
Concrete. Glass fibers used in concrete, 429(C)
Control systems. On guidance and control requirements in astronautics (Roberson), 196
Second-order non-linear difference equations containing small parameters (Torng), 97
Tiny tunnel diode, 161(C)
Corrosion. Non-corrosive anti-freeze, 353 (C)
Creep. A further creep analysis for rotating solid disks of variable thickness (Ma), 408
Cresson Medal. Hugh Latimer Dryden, 146
Cresson Medal Fund. *See* page 144
Cylindrical shells. Comparison of yield conditions for circular cylindrical shells (Sawszuk and Hodge), 362

D

Data processing. Air Reservists' statistics turned over to electronic data-processing system, 77(C)
Microwave data processing, 75(C)
Moderate-price computer, 504(C)
Solid state data processing system, 432(C)
World-wide multi-purpose transmission system, 498(C).
Decontamination. Remote control fallout sweeper, 251(C)
Defocus characteristics. Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184
Delay lines. Delay-line secondary responses in AM and FM sweep integrators (Urkowitz), 1
Detectors. New scintillation detector preamplifier 255(C)
Dielectric lenses. Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184
Difference equations. Second-order non-linear difference equations containing small parameters (Torng), 97
Diodes. High-speed silicon pnpn diode switch, 433(C)
Tiny tunnel diode, 161(C)
"Varactor" diode used in radio receiver, 157(C)
Disks. A further creep analysis for rotating solid disks of variable thickness (Ma), 408
Distillation. Aluminum salt water distillation unit, 351(C)
Dryden, Hugh Latimer. Cresson Medal, 146
Dyeing. Leveling agent improves dyeing, 429(C)
Dynamic response of large complex systems (Mesarović), 274

E

Economics. Are we confused about America's direction? (Romney), 133
Elasticity. Some considerations of the mechanics and design limitations of autofrettage (Faupel), 474
Electrochemistry. Fuel cells (Liebhafsky and Niedrach), 257
Electromagnetics. The thunderstorm as a chemical phenomenon (Konow), 439
Electromechanics. The ferreed, 430(C)
Electron tubes. Continuous-production vacuum furnace, 436(N)
Tiny new electron tube, 354(C)
Electronics. Air Reservists' statistics turned over to electronic data-processing system, 77(C)
Alumina powder used for potting, 160(C)
Electronic survey system, 505(C)
The ferreed, 430(C)
Halogen leak detector for contaminated atmospheres, 255(C)
Moderate-price computer, 504(C)
Photo-optical memory system, 72(C)
"Twistors" for electronic memory devices, 256(C)
U. S. television weather satellite, 500(C)
Engineering psychology. A servomechanisms approach to skill development (Krendel and McRuer), 24

Engines. Plug nozzle engine, 352(C)
Ethics. Our unbalanced biad (Leake), 355

F

Fallout. Remote control fallout sweeper, 251(C)
Ferroelectric converter. High-voltage electricity from sunbeams, 158(C)
Finite regions. On Helmholtz's theorem in multiply-bounded and multiply-connected regions (Van Bladel), 445
Flooring. Conductive flooring for hospital operating rooms, 495(N)
Fluid flow. Hydraulic analogue for one dimensional unsteady gas dynamics (Loh), 43
Fluoroscopes. Fluoroscopic screen detects presence of stray radiation, 433(C)
The Franklin Institute
Annual report, 315
Bartol Research Foundation (*See* Bartol Research Foundation in main index)
Committee on Science and the Arts (*See* Committee on Science and the Arts in main index)
Cresson Medal Fund. *See* page 144
Franklin Institute Laboratories (*See* Franklin Institute Laboratories in main index)
History of the Franklin Institute (LePage), 126
Journal of The Franklin Institute (*See* Journal of The Franklin Institute in main index)
Library (*See* Library of The Franklin Institute in main index)
The library of The Franklin Institute (Hepburn), 221
Meetings (*See* Meetings of The Franklin Institute in main index)
Membership (*See* Members of The Franklin Institute in main index)
Proceedings of the Stated Meeting Held November 18, 1959. Presentation of the Vermilye Medal and observance of the 135th anniversary of The Franklin Institute, 125
The Redding Lecture. The science of yesterday, today and tomorrow (Swann), 163
The science of yesterday, today and tomorrow (Swann), 163
Walter, Lenore and Harry, Memorial Scholarship Awarded, 491
Franklin Institute Laboratories
"Lamprey" (Hargens), 147
Producing burns for medical research (Hargens), 60
Fuel cells (Liebhafsky and Niedrach), 257
Functional analysis. On Helmholtz's theorem in multiply-bounded and multiply-connected regions (Van Bladel), 445

G

Gas dynamics. Hydraulic analogue for one dimensional unsteady gas dynamics (Loh), 43
Gas turbines. A further creep analysis for rotating solid disks of variable thickness (Ma), 408
Gaskets. Improved rubber gaskets, 430(C)
Generators. Rocket's exhaust heat turned to electric power, 155(C)
Glass. Glass fibers used in concrete, 429(C)
Gravitation. Satellite perturbations from extra-terrestrial gravitation and radiation pressure (Geyling), 375
A simple unified field equation and hypothesis (Gresky), 105
Guidance. On guidance and control requirements in astronautics (Roberson), 196
Gyro compasses. Calibration of aircraft compass transmitters, 253(C)

H

Heat conversion. High-voltage electricity from sunbeams, 158(C)
Rocket's exhaust heat turned to electric power, 155(C)
Helmholtz's theorem. On Helmholtz's theorem in multiply-bounded and multiply-connected regions (Van Bladel), 445
Hill, Richard. Lenore and Harry Walter Memorial Scholarship Awarded, 491
History. History of The Franklin Institute (LePage), 126
The library of The Franklin Institute (Hepburn), 221
Hoop stresses. Stresses in certain thin plates rotating about an axis lying in their middle plane (Dhaliwal), 463
Hospitals. Conductive flooring for hospital operating rooms, 495(N)

Humanities. Our unbalanced biad (Leake), 355
Hydraulic analogue for one dimensional unsteady gas dynamics (Loh), 43
Hydrox fuel cells. Fuel cells (Liebhafsky and Niedrach), 257

I

Image intensifiers. Instrument magnifies light 50,000 times, 156(C)
Impedance. The accuracy of impedance measurements in a.c. bridges (Wolff), 299
Industrial management. Are we confused about America's direction? (Romney), 133
Instruments. Advanced oscilloscope, 503 (C)
Calibration of aircraft compass transmitters, 253(C)
Instrument magnifies light 50,000 times, 156(C)
New scintillation detector preamplifier, 255(C)
Transistorized fuel gage, 253(C)
TV optical system for dental medical use, 157(C)
Ion-exchange membranes. Fuel cells (Liebhafsky and Niedrach), 257

J

Jet engines. A further creep analysis for rotating solid disks of variable thickness (Ma), 408
Journal of The Franklin Institute
Publications received: 70, 154, 249, 425, 494

K

Keidel, Frederick A.: Longstreth Medal, 234
Kinematics. The Thomas precession (Zatzkis), 268

L

Laboratory animals. Halogen leak detectors find new use, 353(C)
Leak detectors. Halogen leak detector for contaminated atmospheres, 255(C)
Halogen leak detectors find new use, 353(C)
Lenses. Lens design problems solved by matrix algebra, 497(N)
LePage, Wynn Laurence. Proceedings of the Stated Meeting held November 18, 1959. Presentation of the Vermilye Medal and observance of the 135th anniversary of The Franklin Institute, 125
Letter-sorting machine, 73(C)
Leveling agent improves dyeing, 429(C)
Library of The Franklin Institute
Library of The Franklin Institute (Hepburn), 221
Recent additions: 58
Lightning. The thunderstorms as a chemical phenomenon (Konow), 439
Limit analysis. Comparison of yield conditions for circular cylindrical shells (Sawczuk and Hodge), 362
Logics. Our unbalanced biad (Leake), 355
Longstreth Medal
John W. Coltman, Jr., 57
Frederick A. Keidel, 234
Moulton B. Taylor, 344

M

Magnetism. A simple unified field equation and hypothesis (Gresky), 105
Management. Are we confused about America's direction? (Romney), 133
Masers. A note on the Manley-Rowe energy relations for parametric amplifiers (Callen), 93
Matrix algebra. Lens design problems solved by matrix algebra, 497(N)
Measurement. The accuracy of impedance measurements in a.c. bridges (Wolff), 299
Advanced oscilloscope, 503(C)
Electronic survey system, 505(C)
Medical science. Conductive flooring for hospital operating rooms, 495(N)
Halogen leak detectors find new use, 353 (C)
TV optical system for dental, medical use, 157(C)
Meetings of The Franklin Institute
Minutes: December 16, 1959, 56; January 20, 1960, 143; February 17, 1960, 229; March 16, 1960, 343; April 20, 1960, 420; May 18, 1960, 490
Members of The Franklin Institute
New members: 57, 145, 231, 344, 421
Roll of honor, 231
Silver members for 1959 honored, 230

Memory devices. Photo-optical memory system, 72(C)
"Twistors" for electronic memory devices, 256(C)
Metallurgy. Thermocouple measures temperature of molten metal, 428(C)
Yttrium as an alloying metal, 71(C)
Zone melting of boron, 75(C)
Meteorology. U. S. television weather satellite, 500(C)
Micrometeorites. Tiny microphones record space "hits," 428(C)
Microwave data processing, 75(C)
Military engineering. Airborne crane-shovel, 73(C)
Aluminum salt water distillation unit, 351(C)
Fifteen-story missile tower, 431(C)
Undersea Arctic cable, 434(C)
Miniaturization. Tiny new electron tube, 354(C)
Tiny tunnel diode, 161(C)
Missiles. Fifteen-story missile tower, 431 (C)
Rocket's exhaust heat turned to electric power, 155(C)
Underwater detection system, 350(C)
World-wide transmission system, 498(C)
Mission characteristics. On guidance and control requirements in astronautics (Roberson), 196

N

Nonlinear circuit theory. A note on the Manley-Rowe energy relations for parametric amplifiers (Callen), 93
Nonlinear difference equations. Second-order non-linear difference equations containing small parameters (Torng), 97
Nonlinear oscillation. Oscillating systems with one variable element (Cunningham), 81
Nose cones. Nose cone "bends" to study solar activity, 254(C)
Nuclear emulsion recovery vehicle, 76(C)
Nuvister. Tiny new electron tube, 354(C)

O

Operations analysis. On guidance and control requirements in astronautics (Roberson), 196
Optics. Instrument magnified light 50,000 times, 156(C)
Photo-optical memory system, 72(C)
Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184
TV optical system for dental, medical use, 157(C)
Orbits. Satellite perturbations from extra-terrestrial gravitation and radiation pressure (Geyling), 375
Oscillation. Oscillating systems with one variable element (Cunningham), 81
Oscilloscopes. Advanced oscilloscope, 503 (C)

P

Paper. "Magic paper" for guarding secrecy of facsimile mail, 80(C)
Parametric amplifiers. A note on the Manley-Rowe energy relations for parametric amplifiers (Callen), 93
Parametric oscillation. Oscillating systems with one variable element (Cunningham), 81
Personnel records. Air Reservists' statistics turned over to electronic data-processing system, 77(C)
Perturbation. Satellite perturbations from extra-terrestrial gravitation and radiation pressure (Geyling), 375
Philadelphia history. The library of The Franklin Institute (Hepburn), 221
Philosophy. The science of yesterday, today and tomorrow (Swann), 163
Photography. Photopolymer printing plates, 251(C)
Photopolymer printing plates, 251(C)
Physical laws. A simple unified field equation and hypothesis (Gresky), 105
Plasticity. Comparison of yield conditions for circular cylindrical shells (Sawczuk and Hodge), 362
Plastics. Glass fibers used in concrete, 429(C)
Photopolymer printing plates, 251(C)
Plug nozzle engine, 352(C)
Post office. Letter-sorting machine, 73(C)
"Magic paper" for guarding secrecy of facsimile mail, 80(C)
Potting. Alumina powder used for potting, 160(C)

Preamplifiers. New scintillation detector preamplifier, 255(C)
Printing. Photopolymer printing plates, 251(C)
Probes. Nose cone "bends" to study solar activity, 254(C)
Nuclear emulsion recovery vehicle, 76(C)
Propellants. Safer production of solid propellants, 159(C)
Transistorized fuel gage, 253(C)
Propulsion. Hydraulic analogue for one dimensional unsteady gas dynamics (Loh), 43
Plug nozzle engine, 352(C)

Q

Quasi-linear process. A servomechanisms approach to skill development (Krendel and McRuer), 24

R

Radio. Design of corner reflector antennas, 437(N)
Double-emitter multi-purpose transistor, 498(C)
Radio system for warning drivers, 503(C)
Sun as relay for radio signals, 502(C)
"Varactor" diode used in radio receiver, 157(C)
Radioactivity. Remote control fallout sweeper, 251(C)
Removal of atomic waste from water supplies, 499(C)
Radiology. Fluoroscopic screen detects presence of stray radiation, 433(C)
Redding Lecture. The science of yesterday, today and tomorrow (Swann), 163
Re-entry. Nuclear emulsion recovery vehicle, 76(C)
Underwater detection system, 350(C)
Reflectors. Design of corner reflector antennas, 437(N)
Refracting surfaces. Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184
Relativity. A simple unified field equation and hypothesis (Gresky), 105
The Thomas precession (Zatzkis), 268
Relay. Sun as relay for radio signals, 502(C)
Remote control fallout sweeper, 251(C)
Rockets. New facts on solar x-rays, 78(C)
Plug nozzle engine, 352(C)
Rocket's exhaust heat turned to electric power, 155(C)
Safer production of solid propellants, 159(C)
Romney, George. Proceedings of the Stated Meeting held November 18, 1959. Presentation of the Vermilye Medal and observance of the 135th anniversary of The Franklin Institute, 125
Rubber. Improved rubber gaskets, 430(C)

S

Safety engineering. Fluoroscopic screen detects presence of stray radiation, 433(C)
Radio system for warning drivers, 503(C)
Safer production of solid propellants, 159(C)
Salt water. Aluminum salt water distillation unit, 351(C)
Sampled-data control systems. Second-order non-linear difference equations containing small parameters (Torng), 97
Satellites. High-voltage electricity from sunbeams, 158(C)
Satellite perturbations from extraterrestrial gravitation and radiation pressure (Geyling), 375
Tiny microphones record space "hits," 428
U. S. television weather satellite, 500(C)
Science. Our unbalanced biad (Leake), 355
The science of yesterday, today and tomorrow (Swann), 163
Sealing. Continuous-production vacuum furnace, 436(N)
Improved rubber gaskets, 430(C)
Secondary responses. Delay-line secondary responses in AM and FM sweep integrators (Urkowitz), 1
Semiconductors. Double-emitter multi-purpose transistor, 498(C)
Servomechanisms. A servomechanisms approach to skill development (Krendel and McRuer), 24
Shells. Comparison of yield conditions for circular cylindrical shells (Sawczuk and Hodge), 362
Some considerations of the mechanics and design limitations of autofrettage (Faupel), 474

Signal integrators. Delay-line secondary responses in AM and FM sweep integrators (Urkowitz), 1
Silicon. High-speed silicon pnpn diode switch, 433(C)
Solar activity. Nose cone "bends" to study solar activity, 254(C)
Solar flares. New facts on solar x-rays, 78(C)
Solar power. High-voltage electricity from sunbeams, 158(C)
Sunshine powers air conditioners, 160(C)
Solid propellants. Safer production of solid propellants, 159(C)
Solid state. Solid state data processing system, 432(C)
Space research. Nose cone "bends" to study solar activity, 254(C)
Tiny microphones record space "hits," 428(C)
Steel. Yttrium as an alloying metal, 71(C)
Stochastic processes. A servomechanisms approach to skill development (Krendel and McRuer), 24
Strain. Some considerations of the mechanics and design limitations of autofrettage (Faupel), 474
Stress. Comparison of yield conditions for circular cylindrical shells (Sawczuk and Hodge), 362
A further creep analysis for rotating solid disks of variable thickness (Ma), 408
Some considerations of the mechanics and design limitations of autofrettage (Faupel), 474
Stresses in certain thin plates rotating about an axis lying in their middle plane (Dhaliwal), 463
Structures. Glass fibers used in concrete, 429(C)
Sun. Sun as relay for radio signals, 502(C)
Surgery. Conductive flooring for hospital operating rooms, 495(N)
Surveying. Electronic survey system, 505(C)
Sweep integrators. Delay-line secondary responses in AM and FM sweep integrators (Urkowitz), 1
Switches. The ferreed, 430(C)
High-speed silicon pnpn diode switch, 433(C)
Switching. Photo-optical memory system, 72(C)
Systems engineering. Dynamic response of large complex systems (Mesarović), 274
A servomechanisms approach to skill development (Krendel and McRuer), 24

T

Taylor, Moulton B. Longstreth Medal, 344
Telephony. The ferreed, 430(C)
Television. TV optical system for dental, medical use, 157(C)
U. S. television weather satellite, 500(C)
Temperature. Thermocouple measures temperatures of molten metal, 428(C)
Textiles. Leveling agent improves dyeing, 429(C)
Theory of systems (Mesarović), 274
Thermocouples. Thermocouple measures temperatures of molten metal, 428(C)
Thin plates. Stresses in certain thin plates rotating about an axis lying in their middle plane (Dhaliwal), 463
Thomas precession (Zatzkis), 268
Thunderstorms. The thunderstorm as a chemical phenomenon (Konow), 439
Time-varying oscillation. Oscillating systems with one variable element (Cunningham), 81
Traffic. Radio system for warning drivers, 503(C)
Transistors. Double-emitter multi-purpose transistor, 498(C)
Transistorized fuel gage, 253(C)
Transmission. Delay-line secondary responses in AM and FM sweep integrators (Urkowitz), 1
Sun as relay for radio signals, 502(C)
World-wide transmission system, 498(C)

U

Underwater detection system, 350(C)
Unified field theory. A simple unified field equation and hypothesis (Gresky), 105
Unsteady flow mechanics. Hydraulic analogue for one dimensional unsteady gas dynamics (Loh), 43

V

Vacuum furnaces. Continuous-production vacuum furnace, 436(N)

Vermilye Medal. Are we confused about America's direction (Romney), 133
Proceedings of the Stated Meeting held November 18, 1959. Presentation of the Vermilye Medal and observance of the 135th anniversary of The Franklin Institute, 125

W

Warning systems. Radio system for warning drivers, 503(C)
Undersea Arctic cable, 434(C)
Wastes. Removal of atomic wastes from water supplies, 499(C)
Water treatment. Removal of atomic waste from water supplies, 499(C)
Wavefronts. Relations concerning refracting surfaces, wavefronts, and phase errors (Cheng), 184
Weather. U. S. television weather satellite, 500(C)
Wheatstone bridges. The accuracy of impedance measurements in a.c. bridges (Wolff), 299

X

X-rays. New facts on solar x-rays, 78(C)

Y

Yttrium as an alloying metal, 71(C)

Z

Zone melting of boron, 75(C)

AUTHOR INDEX

Full titles of papers are listed only under first authors; all co-authors are cross-referenced to the full entry. Entries with (B) indicate Bartol Research Foundation Notes; those with (F), Franklin Institute Laboratories Notes.

Agarwal, S. P.: *See* Pomerantz, M. A., S. P. Agarwal and V. R. Potnis

Callen, Herbert B.: A note on the Manley-Rowe energy relations for parametric amplifiers, 93

Cheng, David K.: Relations concerning refracting surfaces, wavefronts, and phase errors, 184

Cunningham, W. J.: Oscillating systems with one variable element, 81

Dhaliwal, R. S.: Stresses in certain thin plates rotating about an axis lying in their middle plane, 463

Faupel, J. H.: Some considerations of the mechanics and design limitations of autofrettage, 474

Geyling, F. T.: Satellite perturbations from extra-terrestrial gravitation and radiation pressure, 375

Gresky, A. T.: A simple unified field equation and hypothesis, 105

Hargens, C. W.: "Lamprey," 147(F)
Producing burns for medical research, 60 (F)

Hepburn, Joseph Samuel: The library of The Franklin Institute, 221

Hodge, Philip G., Jr.: *See* Sawczuk, Antoni and Philip G. Hodge, Jr.

Konow, R. v.: The thunderstorm as a chemical phenomenon, 439

Krendel, Ezra S. and Duane T. McRuer: A servomechanisms approach to skill development, 24

Leake, Chauncey D.: Our unbalanced biad, 355

LePage, Wynn Laurence: History of The Franklin Institute, 126

Liebhafsky, Herman A. and Leonard W. Niedrach: Fuel cells, 257

Loh, W. H. T.: Hydraulic analogue for one dimensional unsteady gas dynamics, 43

Ma, B. M.: A further creep analysis for rotating solid disks of variable thickness, 408

McRuer, Dunae T.: *See* Krendel, Ezra S. and Duane T. McRuer

Mesarović, Mihajlo D.: Dynamic response of large complex systems, 274

Niedrach, Leonard W.: *See* Liebhafsky, Herman A. and Leondard W. Niedrach

Pomerantz, M. A., S. P. Agarwal and V. R. Potnis: Balloon flight investigations of primary cosmic rays during solar disturbances, 235(B)

Potnis, V. R.: *See* Pomerantz, M. A., S. P. Agarwal and V. R. Potnis

Roberson, Robert E.: On guidance and control requirements in astronautics, 196

Romney, George: Are we confused about America's direction?, 133

Sawczuk, Antoni and Philip G. Hodge, Jr.: Comparison of yield conditions for circular cylindrical shells, 362

Swann, W. F. G.: The science of yesterday, today and tomorrow, 163

Torng, H. C.: Second-order non-linear difference equations containing small parameters, 97

Urkowitz, Harry: Delay-line secondary responses in AM and FM sweep integrators, 1

Van Bladel, J.: On Helmholtz's theorem in multiply-bounded and multiply-connected regions, 445

Wolff, Hanns H.: The accuracy of impedance measurements in a.c. bridges, 299

Zatzkis, Henry: The Thomas precession, 268

Journal
of
The Franklin Institute

DEVOTED TO

SCIENCE AND THE MECHANIC ARTS

MANAGING EDITOR, NANCY S. GLENN

ASSOCIATE EDITORS:

CHARLES B. BAZZONI
WILLIAM H. BOGHOSIAN
CHARLES L. CRITCHFIELD
RUPEN EKSERGIAN
VIRGIL MORING FAIRES
PAUL D. FOOTE
WALTER C. JOHNSON
ROBERT S. LIVINGSTON
C. E. K. MEES
F. J. MURRAY
W. F. G. SWANN
HUGH S. TAYLOR
MERLE A. TUVE

ARTHUR L. DAY, *Emeritus*

COMMITTEE ON PUBLICATIONS:

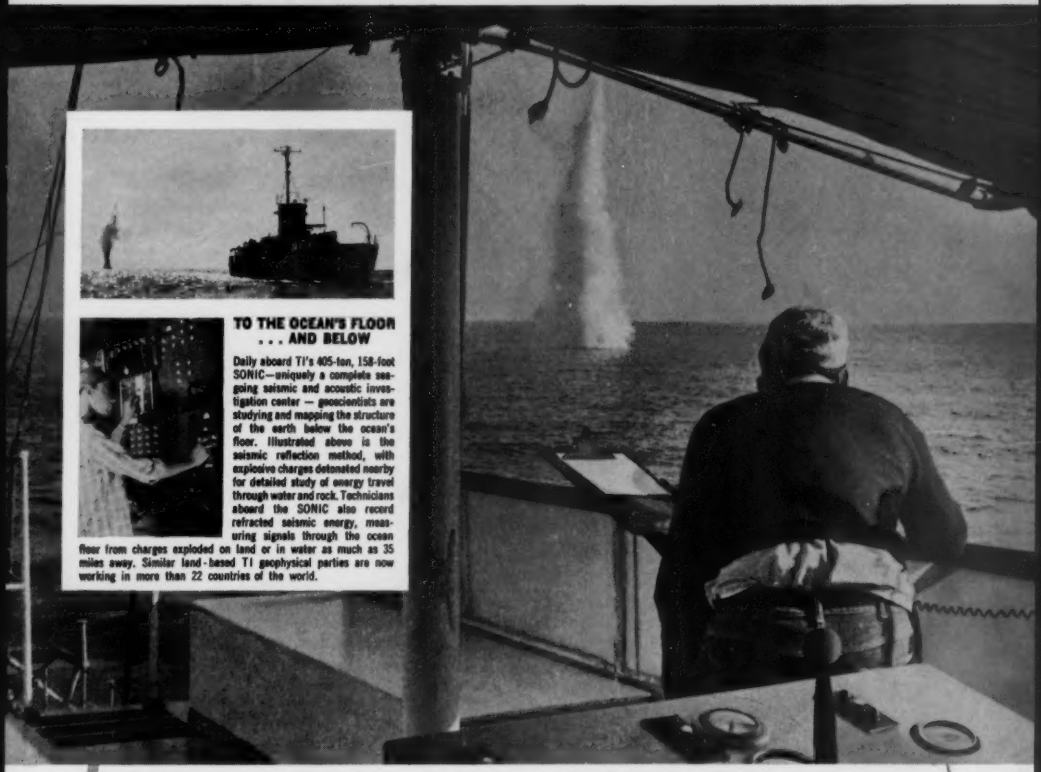
HENRY B. ALLEN
LIONEL F. LEVY, CHAIRMAN
CLARENCE L. JORDAN
PHILIP H. WARD, JR.

VOL. 269—Nos. 1609–1614
(135th Year)

JANUARY–JUNE, 1960

PHILADELPHIA
THE FRANKLIN INSTITUTE
Benjamin Franklin Parkway at 20th Street
1960

LANCASTER PRESS, INC., LANCASTER, PA.



TO THE OCEAN'S FLOOR . . . AND BELOW

Daily aboard TI's 405-ton, 158-foot SONIC—uniquely a complete sea-going seismic and acoustic investigation center — geoscientists are studying and mapping the structure of the earth below the ocean's floor. Illustrated above is the seismic reflection method, with explosive charges detonated nearby for detailed study of energy travel through water and rock. Technicians aboard the SONIC also record refracted seismic energy, measuring signals through the ocean floor from charges exploded on land or in water as much as 35 miles away. Similar land-based TI geophysical parties are now working in more than 22 countries of the world.

INNER-SPACE KNOWLEDGE FOR OUTER-SPACE TECHNOLOGY ADVANCED BY TI'S

scientific key to Davy Jones' locker!

At a time when the focus is on outer space — a time when we know more about the moon's surface than we know about 70% of the earth's surface — geophysicists such as this sea-going scientist from Texas Instruments are furthering space-age technology with studies of inner space — the oceans.

He represents one of more than 60 global TI geophysical investigation parties who are applying 30 years' experience in earth sciences to measurement of earth's land and sea physical characteristics. Fundamental to modern defense, earth sciences have extended their traditional role in petroleum exploration to gathering information vital to programs in *underwater warfare, missile accuracy, seismic communications, detection/surveillance of nuclear tests, and design of underground defense structures.*

TI's talent for these "down-to-earth" studies dates back to 1930, when the company was formed as Geophysical Service Inc. — the first independent company to perform seismic reflection surveys for oil exploration. And from GSI's early need for sensitive electronic seismic equipment came an engineering skill that was applied to Anti-Submarine Warfare systems in World War II, and resulted in TI's Apparatus division becoming a leading ASW equipment manufacturer today. This rare blending of earth sciences knowledge and experience with skills in systems and components gives TI a *unique capability in space-age technology.*

The founding company (GSI) is now the geophysical exploration arm of TI's Geosciences & Instrumentation division, and these capabilities are extended to governmental agencies by the Geosciences department. Designing and manufacturing advanced instruments and systems in wide use by these parties and others is the Instrumentation Product group, with such products as *WORDEN** gravity meters, *EXPLORER** seismograph systems and *seisMAC®* seismic computers.

For more about geosciences at TI, write to Central Merchandising for Bulletin DM-101.

**Trademark of Texas Instruments*

TEXAS INSTRUMENTS
INCORPORATED

HEADQUARTERS: 6000 LEMMON AVENUE, DALLAS 9, TEXAS. **PLANTS:** ATTLEBORO, MASS. • DALLAS, TEXAS • HOUSTON, TEXAS • VERSAILLES, KY. • ELIZABETH SOUTH, AUSTRALIA • ALMELO, HOLLAND • AVERSA, ITALY BEDFORD, ENGLAND • BONNEVILLE, FRANCE • BUENOS AIRES, ARGENTINA • MEXICO CITY, MEXICO

OFFICES IN 75 PRINCIPAL CITIES OF THE WORLD

THE FRANKLIN INSTITUTE LABORATORIES

FOR RESEARCH AND DEVELOPMENT

In the fields of engineering and the physical sciences, a competent and versatile staff of several hundred scientists and engineers, working with modern equipment in a creative climate, is trained to bring a fresh scientific approach to the solution of industrial problems.

THE FRANKLIN INSTITUTE OF THE STATE OF PENNSYLVANIA

BENJAMIN FRANKLIN PARKWAY AT 20TH ST. PHILADELPHIA 3, PA.